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DIRAC EQUATION WITH TWO MASS PARAMETERS

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The Dirac equation is modified in order to allow for two mass parameters (called m_R and m_L). The massless case is especially interesting and is obtained when the product of the two mass parameters is null. In particular, the physical neutrino current can be derived in a more consistent manner than usual. Some possible applications are discussed.

1. *Introduction*

This paper presents a modified Dirac equation which allows for two scalar mass parameters (called m_R and m_L). The outlined treatment is done before second quantization. For massless particles ($m_R m_L = 0$), the case with $m_R = 0$ and $m_L \neq 0$ is well–suited for the description of neutrinos, since the corresponding left–handed current is conserved in general. The same is not true for the right–handed current which is, however, physically unobservable. Notation is rather standard throughout the paper. In particular, and unless otherwise noted, Greek (Latin) indices run through the values 0,1,2,3 (1,2,3) and the summation convention is applied to repeated up and down labels. Units are such that $\hbar = c = 1$.

Some applications are discussed in §5 and §6. A 12–dimensional spinoral formalism is introduced in §5. This affords a compact description of Dirac particles with three possible mass states (massive, left–handed massless and tachyonic). In §6, massless states of two different types are combined into an 8–dimensional spinoral formalism: this is suited for the study of the electroweak interaction.

2. Dirac equation

In a frame of reference $\mathcal{X}$ of real space-time coordinates $x = \{x^\mu\}$ and pseudoeuclidean metric $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, the Dirac equation may be written as follows

$$\mathrm{i}\gamma^\beta \partial_\beta \Psi(x) = m\Psi(x), \quad m \geq 0. \tag{1}$$

Here, Ψ is a complex four–spinor and the Dirac matrices γ^μ (in a fixed chosen representation) obey the usual rules [1–9]

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I, \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \tag{2}$$

with I being the 4×4 identity matrix. Thus, the Ψ solutions of Eq. (1) are eigenstates of the squared four–momentum operator $\mathcal{P}^2 = \mathrm{i}\partial_\alpha g^{\alpha\beta}\mathrm{i}\partial_\beta$ with eigenvalue m^2. For convenience, we further define the Dirac matrix $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$; it is hermitian and unitary, and anticommutes with all γ^μ. Also, we remark the alternative space-time notation: $x = \{t, s\}$ with $t = x^0$ and $s = \{x^k\}$.

Equation (1) can be rewritten in terms of right–handed and left–handed spinors. To that end, we first introduce the space–index S and the time–index T of the frame $\mathcal{X}$. Namely [10], $S = 0$ if $\{x^k\}$ is a right–handed triplet ($S = 1$ otherwise) and $T = 0$ if t runs forward ($T = 1$ otherwise). Then

$$\Psi(x) = \Psi_R(x) + \Psi_L(x), \tag{3}$$

where Ψ_R (right–handed component) and Ψ_L (left–handed component) are given by

$$\Psi_R(x) = \frac{1}{2}(I + \epsilon\gamma^5)\Psi(x), \tag{4}$$

$$\Psi_L(x) = \frac{1}{2}(I - \epsilon\gamma^5)\Psi(x), \tag{5}$$

with $\epsilon = (-1)^{S+T}$. Finally, the following system of equations [1–9] is seen to be equivalent to Eq. (1):

$$\mathrm{i}\gamma^\beta \partial_\beta \Psi_R(x) = m\Psi_L(x), \tag{6}$$

$$\mathrm{i}\gamma^\beta \partial_\beta \Psi_L(x) = m\Psi_R(x). \tag{7}$$

In the next section, we will examine a possible modification of Eqs. (6) and (7). This will be especially interesting for the case of massless particles.

3. *Modified Dirac equation*

The conventional Dirac theory of §2 can be easily generalized as follows

$$i\gamma^\beta \partial_\beta \Psi_R(x) = m_L \Psi_L(x), \tag{8}$$

$$i\gamma^\beta \partial_\beta \Psi_L(x) = m_R \Psi_R(x), \tag{9}$$

with m_R, $m_L \in \mathbf{R}$. Equivalently,

$$i\gamma^\beta \partial_\beta \Psi(x) = \frac{1}{2}[(m_R + m_L)I + \epsilon(m_R - m_L)\gamma^5]\Psi(x) \tag{10}$$

so that the Ψ solutions are eigenstates of $\mathcal{P}^2$ with eigenvalue $m_R m_L$. Except for tachyons [11] ($m_R m_L < 0$), there are four possibilities: (i) $m_r = m_L = 0$ (massless); (ii) $m_R m_L > 0$ (massive); (iii) $m_R = 0$ and $m_L \neq 0$ (massless); (iv) $m_R \neq 0$ and $m_L = 0$ (massless).

The case (i) is trivial, as Eq. (10) reduces to Eq. (1) with $m = 0$. The case (ii) can be handled as follows. Define:

$$\Delta^\pm = \frac{1}{2\sqrt{m_R m_L}}[(m_R + m_L)I \pm \epsilon(m_R - m_L)\gamma^5] \tag{11}$$

and notice that Δ^- is the inverse of Δ^+ (and viceversa). Equation (10) reads

$$i\gamma^\beta \partial_\beta \Psi(x) = m\Delta^+ \Psi(x), \quad m = \sqrt{m_R m_L} > 0, \tag{12}$$

and reduces to

$$i\mathring{\gamma}^\beta \partial_\beta \Psi(x) = m\Psi(x), \tag{13}$$

with $\mathring{\gamma}^\mu = \Delta^- \gamma^\mu = \gamma^\mu \Delta^+$ and $\mathring{\gamma}^\mu \mathring{\gamma}^\nu + \mathring{\gamma}^\nu \mathring{\gamma}^\mu = 2g^{\mu\nu} I$. Due to the preceding anticommutation relations, an invertible matrix Λ exist such that [1–9]

$$\Psi(x) = \Lambda \Phi(x), \tag{14}$$

$$i\gamma^\beta \partial_\beta \Phi(x) = m\Phi(x). \tag{15}$$

Therefore, Eq. (10) is solved in terms of a standard massive Dirac equation. In particular, the conserved current may be written as

$$\Phi^\dagger(x)\gamma^0\gamma^\mu\Phi(x). \tag{16}$$

In the cases (iii) and (iv), the above procedure cannot be repeated. In fact, the right–hand–side of Eq. (10) involves $(I \mp \epsilon\gamma^5)$, which are singular. Specifically for (iii), one has

$$i\gamma^\beta \partial_\beta \Psi(x) = \frac{m_L}{2}(I - \epsilon\gamma^5)\Psi(x), \tag{17}$$

that is

$$\Psi_L(x) = \mathrm{i}\gamma^\beta \partial_\beta \left[\frac{\Psi_R(x)}{m_L} \right], \tag{18}$$

$$\mathrm{i}\gamma^\beta \partial_\beta \Psi_L(x) = 0. \tag{19}$$

In general (i.e., for all solutions Ψ of Eq. (17)), the left–handed current

$$[\Psi_L(x)]^\dagger \gamma^0 \gamma^\mu \Psi_L(x) = \Psi^\dagger(x) \gamma^0 \gamma^\mu \left[\frac{1}{2}(I - \epsilon\gamma^5) \right] \Psi(x) \tag{20}$$

is conserved, while the corresponding right–handed current is not. By contrast, both currents are conserved in the massless case of Eq. (1): then, the conserved right–handed current is typically discarded [8] in order to reproduce the experimental evidence for neutrinos. From a theoretical standpoint, discarding a conserved current is a rather disturbing ad hoc procedure. This is not required here and $\Psi_R(x)/m_L$ can be interpreted as a physically unobservable "potential" for Ψ_L (Eq. (18)), while Ψ_L obeys the basic equations of massless left–handed Weyl–type neutrinos (Eq. (19) and its consequence Eq. (20)). Besides, it is manifest that the outlined approach is substantially different from that of Majorana [1–9].

4. *Tachyonic case*

The tachyonic case ($m_R m_L < 0$) can be handled similarly to the massive case. Specifically, define:

$$\Xi^\pm = -\frac{1}{2\sqrt{|m_R m_L|}}[(m_R - m_L)I \pm \epsilon(m_R + m_L)\gamma^5], \tag{21}$$

and notice that Ξ^- is the inverse of Ξ^+ (and viceversa). Equation (10) reads

$$\mathrm{i}\gamma^\beta \partial_\beta \Psi(x) = -\epsilon m \Xi^+ \gamma^5 \Psi(x), \quad m = \sqrt{|m_R m_L|} > 0, \tag{22}$$

and reduces to

$$\mathrm{i}\tilde{\gamma}^\beta \partial_\beta \Psi(x) = -\epsilon m \gamma^5 \Psi(x), \tag{23}$$

with $\tilde{\gamma}^\mu = \Xi^- \gamma^\mu = \gamma^\mu \Xi^+$ and $\tilde{\gamma}^\mu \tilde{\gamma}^\nu + \tilde{\gamma}^\nu \tilde{\gamma}^\mu = 2g^{\mu\nu} I$. Also: $\tilde{\gamma}^5 \equiv \mathrm{i}\tilde{\gamma}^0 \tilde{\gamma}^1 \tilde{\gamma}^2 \tilde{\gamma}^3 = \gamma^5$. Due to preceding relations, an invertible 4×4 matrix Ω exists such that [1–9]

$$\Psi(x) = \Omega\Theta(x), \tag{24}$$

$$\mathrm{i}\gamma^\beta \partial_\beta \Theta(x) = -\epsilon m \gamma^5 \Theta(x). \tag{25}$$

Here, Eq. (25) can be taken as a possible standard form of the tachyonic Dirac equation [11]. The related conserved current may be conveniently written as

$$-\epsilon \Theta^\dagger(x) \gamma^0 \gamma^\mu \gamma^5 \Theta(x). \tag{26}$$

5. *Mass states*

As noticed in §3, taking $m_L \neq 0$ insures that Eq. (10) can be used to properly describe the physical neutrino current, with no redundancies. If, for this reason, a non–null m_L is imposed in this section, only threee cases are left: the massive case, the left–handed massless case (iii) and the tachyonic case. In the aforementioned order, these cases produce Eqs. (15), (17) and (25), which in turn originate the relevant conserved currents. The present section combines these three equations into a 12–dimensional spinoral formalism.

It is evident that Eqs. (15), (17) and (25) can be rewritten as follows

$$\mathrm{i}\gamma^{\beta}\partial_{\beta}\Upsilon_{\sigma}(x) = \frac{m}{2}[(I - \epsilon\gamma^5) - \sigma(I + \epsilon\gamma^5)]\Upsilon_{\sigma}(x) \tag{27}$$

with $m > 0$ and $\sigma = -1, 0, 1$. Equation (27) describes three mass states (one for each of the listed values of σ: massive, massless and tachyonic, in that order), corresponding to the same value of m. In relationship to Eq. (10), this parameter can be interpreted as $\sqrt{|m_R m_L|}$ for $\sigma = \pm 1$. For $\sigma = 0$, interpretations are unnecessary: in this case, the value of m is essentially irrelevant to begin with, as it does not affect in any ways the left–handed current (see §3 for clarity). At any rate, the standard massless Dirac equation would differ from Eq. (27), in that it would require: $\sigma = 0 \Rightarrow m = 0$.

As already anticipated above, Eq. (27) can be interpreted as one that applies to three different states of the same particle. Hence, it is convenient to modify the formalism as follows. For any 4×4 Dirac matrix γ, define

$$\Gamma = \begin{pmatrix} \gamma & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \gamma \end{pmatrix} \tag{28}$$

where $\mathbf{0}$ is the 4×4 null matrix. Also, introduce the 12–spinor $\boldsymbol{\Upsilon}(\mathbf{x})$ as the column of the three $\Upsilon_{\sigma}(x)$ spinors (placed, from top to bottom, in ascending order of σ). Finally, define

$$\Sigma = \begin{pmatrix} -I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{pmatrix}. \tag{29}$$

Then, Eq. (27) can be cast in the following form:

$$\mathrm{i}\Gamma^{\beta}\partial_{\beta}\boldsymbol{\Upsilon}(x) = m\mathrm{W}\boldsymbol{\Upsilon}(x), \tag{30}$$

where

$$\mathrm{W} = \frac{1}{2}\left[\left(\mathbb{I} - \epsilon\Gamma^5\right) - \left(\mathbb{I} + \epsilon\Gamma^5\right)\Sigma\right], \tag{31}$$

and $\mathbb{I}$ stands for the 12×12 identity matrix.

It is worthwhile remarking that tachyons could be avoided all together, if considered undesirable. This may be done by imposing $m_R m_L \geq 0$ in §3 and modifying the developments of this section accordingly (8–spinors in place of 12–spinors, etc.).

6. Massless states for the electroweak interaction

A modification of the treatment outlined in the previous section may be employed to discuss the electroweak interaction. Consider cases (i) and (iii) of §3 and rewrite them as follows:

$$\mathrm{i}\gamma^\beta \partial_\beta \Upsilon_\sigma(x) = \frac{m}{2}\left[\left(\frac{1}{2}+\sigma\right)\left(I-\epsilon\gamma^5\right)\right]\Upsilon_\sigma(x) \tag{32}$$

with $m > 0$, $\sigma = -1/2$ (case (i)) and $\sigma = +1/2$ (case (iii)). Introduce the 8–dimensional Dirac matrix notatiom

$$\Gamma = \begin{pmatrix} \gamma & \mathbf{0} \\ \mathbf{0} & \gamma \end{pmatrix}, \tag{33}$$

and furthermore:

$$\Sigma = \frac{1}{2}\begin{pmatrix} -I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}, \tag{34}$$

$$\boldsymbol{\Upsilon}(x) = \begin{pmatrix} \Upsilon_{-1/2}(x) \\ \Upsilon_{+1/2}(x) \end{pmatrix}. \tag{35}$$

Cast Eq. (32) in the form:

$$\mathrm{i}\Gamma^\beta \partial_\beta \boldsymbol{\Upsilon}(x) = m\mathrm{W}\boldsymbol{\Upsilon}(x) \tag{36}$$

with

$$\mathrm{W} = \frac{1}{2}\left[\left(\mathbb{I}-\epsilon\Gamma^5\right)\left(\frac{1}{2}\mathbb{I}+\Sigma\right)\right], \tag{37}$$

where $\mathbb{I}$ is the 8×8 identity matrix. Equation (36) now describes two different massless states: a state of type (i) (which generates both a left–handed and a right–handed conserved current) and a state of type (iii) (which only needs the left–handed spinor to be physically described; see §3 for clarity).

The result of the aforementioned approach is that of producing a framework suited for the study of the electroweak interaction. In fact, Eq. (36) implies that a full spinor (i.e., two spinoral components: left–handed and right–handed) be given physical significance for the state of type (i). This is the state (electron) which will acquire mass through the usual symmetry breaking process, and will consequently lose the separate conservation of left–handed and right-handed currents. At the same time, the state of type (iii) can be physically described by its left-handed spinoral component only: this is the state (neutrino) which shall remain massless after symmetry breaking.

By comparison, the usual treatment of the electroweak interaction [1,2,6] starts with an equation like Eq. (36), but with $m = 0$ (both states of type (i)): thus, there are no actual theoretical justifications for intoducing only one spinoral component (left–handed) for the state which remains massless after symmetry breaking. It is, once again, a rather disturbing ad hoc procedure, breaking the symmetry even before the symmetry breaking process is supposed to have started! In our treatment, a distinction arises at the level of equations of motion: a case of type (i) is for particles which are not truly massless, but acquire mass through some symmetry breaking process; a case of type (iii) is for truly massless (left–handed) particles.

In order to verify that Eq. (36) is appropriate for the study of the electroweak interaction, its conserved currents must also be mentioned. Besides the currents trivially implied by case (i) and case (iii) separately, the following "mixing" currents are also conserved:

$$\boldsymbol{\Upsilon}^\dagger(x)\Gamma^0\Gamma^\mu(\mathrm{Q}+\mathrm{Q}^\dagger)\boldsymbol{\Upsilon}(x), \tag{38}$$

$$\mathrm{i}\boldsymbol{\Upsilon}^\dagger(x)\Gamma^0\Gamma^\mu(\mathrm{Q}-\mathrm{Q}^\dagger)\boldsymbol{\Upsilon}(x), \tag{39}$$

where

$$\mathrm{Q} = \begin{pmatrix} \mathbf{0} & \frac{1}{2}(I-\epsilon\gamma^5) \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{40}$$

7. *Conclusions*

As discussed at the end of §3, the standard treatment of massless neutrinos produces an unwanted conserved current, which is discarded. While the procedure (or theoretical justification) for this elimination may be more or less sophisticated, the only honest approach is similar to that of Ref. 8: it is a postulate put forth in order to agree with the experimental evidence. The formulation here outlined gives the possibility of building this agreement directly into the equation of motion (Eq. (17)), without having to repeat Majorana's treatment.

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DIRACOVA JEDNADŽBA S DVA MASENA PARAMETRA

Proširenje Diracove jednadžbe omogućuje uvođenje dva parametra za masu (m_R i m_L). Posebno je zanimljiv bezmaseni slučaj koji se dobiva kada je umnožak dvaju masenih parametara jednak nuli. Ovim se pristupom može izvesti neutrinska struja na prikladniji način. Raspravljaju se moguće primjene.