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NEW SOLUTIONS IN ϕ^4–THEORY WITH DAMPING

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Two-kink and kink-antikink states in ϕ^4-theory with damping are constructed.

1. *Introduction and formulation of the problem*

The ϕ^4-theory has been used for the description of a large variety of physical phenomena [1-3]. It is a nonlinear model that has been thoroughly investigated. When the necessity of description of real processes has appeared, it was necessary to take into account such aspects of the problem as friction and damping, which are always present in physical systems. In the ϕ^4-theory, damping is taken into account by introducing the term proportional to ϕ_t in equation of motion:

$$\phi_{tt} - \phi_{xx} + \alpha\phi_t - \phi + \phi^3 = 0 \qquad (1)$$

where α is the damping coefficient.

Without a loss of generality, all the other coefficients in Eq. (1) may be taken equal to unity.

When damping is absent ($\alpha = 0$), the ϕ^4-theory is nonintegrable. For the self-similarity case the equation of the theory has the solitary wave solution (kink or antikink). The problem of construction of the exact solution as a coupled state of two kinks (or kink and antikink) has not been solved previously. Results of some numerical experiments point out to the possibility of existence such a quasistable states [4-6], but the results of analytical investigations are at variance with them [7].

When we take damping into account ($\alpha \neq 0$), Eq. (1) for the self-similarity case reduces to the ordinary differential equation possessing the Painleve property [8,9]. Such equation is of an integrable type. Therefore, it is possible to consider the problem of constructing the field configuration corresponding to the coupled states of solitary waves in ϕ^4-theory.

In this paper the explicit expressions for coupled states of two kinks or of kink and antikink in self-similarity case are constructed. It should be noticed that the problem of definition of integrability of a nonlinear partial differential equation requires additional explanations [10,11]. The nonlinear partial differential equation will be considered integrable if it has the N-soliton solution, $N = 1, 2, 3, \ldots$. For such an equation there is an infinite number of conservation laws. So this paper may be considered as the first step in constructing the N-soliton solution for Eq. (1) and hence the proof of its integrability. Besides, the construction of a new, previously unknown solution for Eq. (1) is of general interest.

To construct the solutions corresponding to coupled states mentioned above, the direct integration metod based on the well-known Hirota method is used with some modifications.

Let

$$\phi(x,t) = \sigma \frac{F_x}{F} \tag{2}$$

where $F(x,t)$ is an unknown function and σ is a constant determined below. Substitution Eq. (2) into Eq. (1) results in the following cubic differential equation

$$\begin{aligned} F_{xtt}F^2 - 2F_{xt}F_tF - F_xF_{tt}F + 2F_xF_t^2 - F_{xxx}F^2 + 3F_xF_{xx}F \\ +(\sigma^2 - 2)F_x^3 + \alpha F_{xt}F^2 - \alpha F_xF_tF - F_xF^2 = 0. \end{aligned} \tag{3}$$

Usually, at this stage of solving, in accordance with Hirota method the value of the parameter σ should be chosen. Seemingly, the value $\sigma^2 = 2$ is the most appropriate, but such a choice leads to considerable difficulties. As will be shown below, it is natural to determine σ putting the expression for a single kink into Eq. (1).

Let us represent $F(x,t)$ as a formal series

$$F(x,t) = 1 + \epsilon f_1 + \epsilon^2 f_2 + \ldots \tag{4}$$

where $f_i(x,t)$ are unknown functions ($i = 1, 2, \ldots$); generally speaking, ϵ is not a small constant. By substituting Eq. (4) into Eq. (3) and equating to zero coefficients for each degree of ϵ, we obtain an infinite system of linear equations that determine $f_i(x,t)$. This system has the form

ϵ^1 :

$$f_{1,xtt} - f_{1,xxx} + \alpha f_{1,xt} - f_{1,x} = 0 \tag{5}$$

ϵ^2 :

$$f_{2,xtt} - f_{2,xxx} + \alpha f_{2,xt} - f_{2,x} = 2f_{1,xt}f_{1,t} + f_{1,x}f_{1,tt}$$
$$-3f_{1,x}f_{1,xx} + \alpha f_{1,x}f_{1,t} \tag{6}$$

ϵ^3 :

$$f_{3,xtt} - f_{3,xxx} + \alpha f_{3,xt} - f_{3,x} = 2f_{1,xt}f_{2,t} + 2f_{1,t}f_{2,xt}$$
$$+f_{1,x}f_{2,tt} + f_{2,x}f_{1,tt} - 2f_{1,x}f_{1,t}^2 - 3f_{1,xx}f_{2,x} - 3f_{1,x}f_{2,xx} - (\sigma^2 - 2)f_{1,x}^3$$
$$+\alpha f_{1,x}f_{2,t} + \alpha f_{1,t}f_{2,x} - f_{2,xtt}f_1 + f_{2,xxx}f_1 - \alpha f_{2,xt}f_1 + f_{2,x}f_1 \tag{7}$$

and so on.

It is clear that for every value i, the function $f_i(x,t)$ will be determined by the previous functions only. The remarkable property of the series (4) is that it contains an finite number of terms for problems of N-soliton solutions. It is not known whether the problem (1) has such a solution for any N. In this paper we demonstrate that for $N = 2$ the series (4) is finite.

2. *One-kink solution*

A one-kink solution for ϕ^4-theory is well-known [8,9,12]. Here it will be constructed again to demonstrate the method of solving. One should use Eq. (5) and look for its solution in the form

$$f_1(x,t) = \mathrm{e}^{\eta_1} \tag{8}$$

where $\eta_1 = k_1 x - \omega_1 t + \eta_1^{(0)}$ and k_1, ω_1 and $\eta_1^{(0)}$ are constants to be determined. By using Eqs. (2), (4) and (5) we obtain

$$\phi(x,t) = \frac{\sigma k_1}{2}\left\{1 + \mathrm{th}[\frac{1}{2}(k_1 x - \omega_1 t + \eta_1^{(0)})]\right\}. \tag{9}$$

From Eq. (5) we can derive the relation between k_1 and ω_1:

$$\omega_1^2 - \alpha\omega_1 - k_1^2 - 1 = 0. \tag{10}$$

The second linearly independent equation to determine k_1 and ω_1 is obtained from the requirement of breaking the series (4) for $i = 2$ in the case of one-kink solution. It means that the right-hand side of Eq. (6) should be equal to zero. This leads to equation

$$3\omega_1^2 - \alpha\omega_1 - 3k_1^2. \tag{11}$$

By solving Eqs. (10) and (11) we obtain

$$\omega_1 = -\frac{3}{2\alpha}, \tag{12}$$

$$k_1^2 = \frac{9 + 2\alpha^2}{4\alpha^2}. \tag{13}$$

Different signs of k_1 obtained from Eq. (13) correspond to the kink and antikink. As it should be, the velocity and the wave number of the kink are fixed.

From Eqs. (1) and (9) we obtain

$$\sigma = \frac{1}{k_1}.$$

For

$$k_1 = -\frac{(9 + 2\alpha^2)1/2}{2\alpha}$$

Eq. (9) yields

$$\phi(x,t) = \frac{1}{2}\left\{1 - \mathrm{th}[\frac{(9 + 2\alpha^2)^{1/2}}{4\alpha}(x - \frac{3t}{(9 + 2\alpha^2)^{1/2}} + \eta_1^{(0)*})]\right\}. \tag{14}$$

This is exactly the solution obtained in Ref. 8. The definition of the constant $\eta_1^{(0)*}$, which describes the initial phase shift, is obvious. The substitution of the solution (8) with parameters ω_1 and k_1 defined by Eqs. (12) and (13) into the right-hand side of Eq. (6) leads to equating it to zero as in the usual Hirota method.

It is clear that taking damping into account leads to non-trivial generalization of the kink in ϕ^4-theory. For $\alpha \to 0$ the solution (14) does not reduce to well-known kink for ϕ^4-theory without damping.

It was pointed out in Ref. 12 that the one-kink solution depends on two arbitrary constants. This conclusion has been criticized in Refs. 8 and 9 where only one arbitrary constant in expression for the one-kink solution was argued. Our analysis agrees with the latter point of view.

3. *Coupled states*

Let us construct explicit expressions for field contributions describing coupled states of two kinks or of kink and antikink in ϕ^4-theory with damping. To do this we shall use Eq. (6).

By substituting the following expression for $f_1(x,t)$

$$f_1(x,t) = \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2}, \tag{15}$$

$\eta_i = k_i x - \omega_i t + \eta_i^{(0)}$, $i = 1,\, 2$, into Eq. (6), we can transform it to the form

$$\begin{aligned} f_{2,xtt} - f_{2,xxx} + \alpha f_{2,xt} - f_{2,x} = (3k_1\omega_1^2 - 3k_1^3 - \alpha k_1\omega_1)\mathrm{e}^{2\eta_1} \\ +[2(k_1+k_2)\omega_1\omega_2 + k_1\omega_2^2 + k_2\omega_1^2 - 3(k_1^2 k_2 + k_1 k_2^2) \\ -\alpha(k_1\omega_2 + k_2\omega_1)]\mathrm{e}^{(\eta_1+\eta_2)} + (3k_2\omega_2^2 - 3k_2^3 - \alpha k_2\omega_2)\mathrm{e}^{2\eta_2}. \end{aligned} \tag{16}$$

Here we don't require the fulfilment of the condition $k_1 \neq k_2$ as in the Hirota method.

We shall look for the solution of Eq. (16) in the form

$$f_2(x,t) = R\,\mathrm{e}^{2\eta_1} + Q\,\mathrm{e}^{\eta_1+\eta_2} + S\,\mathrm{e}^{2\eta_2}. \tag{17}$$

Here R, Q and S are constants to be determined. By substituting Eq. (17) into Eq. (16) and equating to zero coefficients for the same powers we obtain

$$2(4\omega_1^2 - 4k_1^2 - 2\alpha\omega_1 - 1)R = 3\omega_1^2 - 3k_1^2 - \alpha\omega_1, \tag{18}$$

$$2(4\omega_2^2 - 4k_2^2 - 2\alpha\omega_2 - 1)S = 3\omega_2^2 - 3k_2^2 - \alpha\omega_2, \tag{19}$$

$$\begin{aligned} (k_1+k_2)[(\omega_1+\omega_2)^2 - (k_1+k_2)^2 - \alpha(\omega_1+\omega_2) - 1]Q = \\ 2(k_1+k_2)\omega_1\omega_2 + k_1\omega_2^2 + k_2\omega_1^2 - 3(k_1^2 k_2 + k_1 k_2^2) - \alpha(k_1\omega_2 + k_2\omega_1). \end{aligned} \tag{20}$$

Although the parameters k_1 and ω_1 are known, it is impossible to calculate coefficient R from Eq. (18), since this equation leads to the uncertainty of the type 0/0. The equation of coupling between k_1 and ω_1 has yet to be constructed. To determine the coefficients R, Q and S we proceed as for one-kink solution. Let us substitute Eqs. (15) and (17) into the righ-hand side of Eq. (7) and set it equal to zero. We obtain the following coupled equations:

$$(10\omega_1^2 - 10k_1^2 + 2)R = 2\omega_1^2 - 2k_1^2 + 1, \tag{21}$$

$$(10\omega_2^2 - 10k_2^2 + 2)S = 2\omega_2^2 + (\frac{1}{k_1^2} - 2)k_2^2, \tag{22}$$

$$[2k_1\omega_1(\omega_1+\omega_2)+2\omega_1(k_1+k_2)(\omega_1+\omega_2)+k_1(\omega_1+\omega_2)^2$$

$$+(k_1+k_2)\omega_1^2-3k_1^2(k_1+k_2)-3k_1(k_1+k_2)^2-\alpha k_1(\omega_1+\omega_2)$$

$$-\alpha\omega_1(k_1+k_2)-(k_1+k_2)(\omega_1+\omega_2)^2+(k_1+k_2)^3+\alpha(k_1+k_2)(\omega_1+\omega_2)+(k_1+k_2)]Q \quad (23)$$

$$+[4k_2\omega_1\omega_2+8k_1\omega_1\omega_2+4k_2\omega_1^2+2k_1\omega_2^2-6k_2^2k_1-12k_2k_1^2-2\alpha\omega_1k_2$$

$$-2\alpha\omega_2k_1-8k_1\omega_1^2+8k_1^3+4\alpha\omega_1k_1+2k_1]R-4k_1\omega_1\omega_2-2k_2\omega_1^2-3(\frac{1}{k_1^2}-2)k_1^2k_2=0,$$

$$[2k_2\omega_2(\omega_1+\omega_2)+2\omega_2(k_1+k_2)(\omega_1+\omega_2)+k_2(\omega_1+\omega_2)^2$$

$$+(k_1+k_2)\omega_2^2-3k_2^2(k_1+k_2)-3k_2(k_1+k_2)^2-\alpha k_2(\omega_1+\omega_2)$$

$$-\alpha\omega_2(k_1+k_2)-(k_1+k_2)(\omega_1+\omega_2)^2+(k_1+k_2)^3+\alpha(k_1+k_2)(\omega_1+\omega_2)^2+(k_1+k_2)]Q \quad (24)$$

$$+[4k_1\omega_1\omega_2+8k_2\omega_1\omega_2+4k_1\omega_2^2+2k_2\omega_1^2-6k_1^2k_2-12k_1k_2^2-2\alpha\omega_2k_1$$

$$-2\alpha\omega_1k_2-8k_2\omega_2^2+8k_2^3+4\alpha\omega_2k_2+2k_2]S-4k_2\omega_1\omega_2-2k_1\omega_2^2-3(\frac{1}{k_1^2}-2)k_1k_2^2=0.$$

From Eq. (21) we obtain $R=0$.

Now we analyze the expressions for $\phi(x,t)$ for coupled states resulting from different relations between k_i and ω_i ($i=1$,2). If

$$k_1^2=k_2^2=\frac{9+2\alpha^2}{4\alpha^2}$$

then ω_1 and ω_2 are determined by Eq. (12).

There are four different cases.

1) $k_1=k_2$, $\omega_1=\omega_2$

In this case Eq. (19) for the coefficient S leads to the uncertainty of the type 0/0. To determine S we use Eq. (22) which leads to $S=0$. In a similar way Eq. (20) leads to the uncertainty of the type 0/0 for coefficient Q. To determine Q we use Eq. (23) which leads to $Q=0$. Consequently in this case $f_2(x,t)=0$ and for $\phi(x,t)$ we obtain

$$\phi(x,t)=\left[\mathrm{e}^{(k_1x-\omega_1t+\eta_1^{(0)})}+\mathrm{e}^{(k_1x-\omega_1t+\eta_2^{(0)})}\right]$$

$$\times\left[1+\mathrm{e}^{(k_1x-\omega_1t+\eta_1^{(0)})}+\mathrm{e}^{(k_1x-\omega_1t+\eta_2^{(0)})}\right]^{-1}. \quad (25)$$

When the initial phase shifts $\eta_i^{(0)}$ $(i = 1{,}2)$ are equal to zero, the only difference between Eq. (25) from Eq. (14) is a constant phase shift of ln 2.

2) $k_1 = -k_2$, $\omega_1 = \omega_2$

In this case $R = S = 0$. To determine the coefficient Q we use Eq. (26) (Eq. (27) leads to the same result). We obtain

$$Q = \frac{6}{6+\alpha^2}.$$

Then

$$f_2(x,t) = \frac{6}{6+\alpha^2} \mathrm{e}^{(-2\omega_1 t + \eta_{12}^{(0)})}$$

where $\eta_{12}^{(0)} = \eta_1^{(0)} + \eta_2^{(0)}$.

Using Eqs. (2) and (4) the final expression for $\phi(x,t)$ in this case may be written in a form

$$\phi(x,t) = (6+\alpha^2)\left[\mathrm{e}^{(k_1 x - \omega_1 t + \eta_1^{(0)})} - \mathrm{e}^{(-k_1 x - \omega_1 t + \eta_2^{(0)})}\right]$$

$$\times \left\{(6+\alpha^2)\left[1 + \mathrm{e}^{(k_1 x - \omega_1 t + \eta_1^{(0)})} + \mathrm{e}^{(-k_1 x - \omega_1 t + \eta_2^{(0)})}\right] + 6\mathrm{e}^{(-2\omega_1 t + \eta_{12}^{(0)})}\right\}^{-1}. \qquad (26)$$

The behaviour of the function defined by Eq. (26) for $\alpha = 0.7$, $\eta_1^{(0)} = \eta_2^{(0)} = -1.5$ and $k_1 = (9+2\alpha^2)^{1/2}/(2\alpha)$ is shown in Fig. 1. Notice that the non-monotonic behaviour of the function in the region $|x| \leq 1$ is conserved in time.

3) $k_1 = k_2$, $\omega_1 = -\omega_2$

In this case Eq. (21) leads to $R = 0$ and from Eq. (22) we obtain $S = 1/4$. It is possible to calculate the coefficient Q from Eq. (20),

$$Q = \frac{6+\alpha^2}{6+2\alpha^2}.$$

So, $f_2(x,t)$ takes the form

$$f_2(x,t) = \frac{6+\alpha^2}{6+2\alpha^2}\mathrm{e}^{(2k_1 x + \eta_{12}^{(0)})} + \frac{1}{4}\mathrm{e}^{2(k_1 x + \omega_1 t + \eta_2^{(0)})}.$$

As for Eq. (26), this allows writting the following expression for the coupled state

$$\phi(x,t) = 2\left\{2(3+\alpha^2)\left[\mathrm{e}^{(k_1 x - \omega_1 t + \eta_1^{(0)})} + \mathrm{e}^{(k_1 x + \omega_1 t + \eta_2^{(0)})}\right]\right.$$

$$\left. + 2(6+\alpha^2)\mathrm{e}^{(2k_1 x + \eta_{12}^{(0)})} + (3+\alpha^2)\mathrm{e}^{2(k_1 x + \omega_1 t + \eta_2^{(0)})}\right\}$$

$$\times \left\{ 4(3+\alpha^2) \left[1 + \mathrm{e}^{(k_1 x - \omega_1 t + \eta_1^{(0)})} + \mathrm{e}^{(k_1 x + \omega_1 t + \eta_2^{(0)})} \right] \right.$$

$$\left. +2(6+\alpha^2)\mathrm{e}^{(2k_1 x + \eta_{12}^{(0)})} + (3+\alpha^2)\mathrm{e}^{2(k_1 x + \omega_1 t + \eta_2^{(0)})} \right\}^{-1} . \tag{27}$$

The behaviour of the function defined by Eq. (27) for α= 0.7, $\eta_1^{(0)} = \eta_2^{(0)}$ = -1.5 and $k_1 = -(9+2\alpha^2)^{1/2}/(2\alpha)$ is shown in Fig. 2. For this function the non-monotonic character in the region $|x| \leq 1$ vanishes in time.

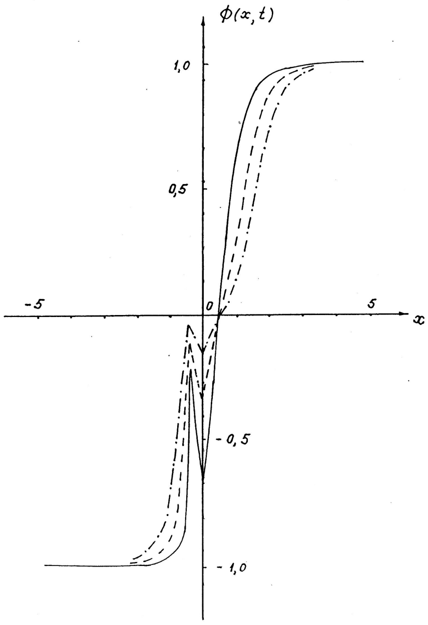

Fig. 1. The behaviour of $\phi(x,t)$ defined by Eq. (26) as function of x for different t: t = 0 - solid line, t = 1 - dashed line and t = 2 dot-dashed line.

4) $k_1 = -k_2$, $\omega_1 = -\omega_2$

It is impossible to determine the coefficient Q in this case. We assume that this case corresponds to a kink and antikink that move apart from the initial position in opposite directions. So the coupled state could not be formulated.

4. Conclusion

In conclusion I would like to make a remark. In the usual Hirota method, the transformation (2) leads to the quadratic type of the differential equation under consideration. The property of the Hirota method is such that an exact truncation in any order equation of the infinite system of linear differential equations guaranties that all higher order equations will also give exactly the vanishing of the further terms in the series of type (4). But in our case Eq. (3) is not quadratic. At present one can not prove whether this property has been retained. Nevertheless, the method of construction the N-soliton solution which is presented here requires the setting equal to zero the right-hand side of the $(N+1)$-soliton equation to obtain the parameters of the N-soliton solution. This is why I think the property of Hirota method mentioned above is fulfilled (at least it is right for one-kink solution and coupled states derived above).

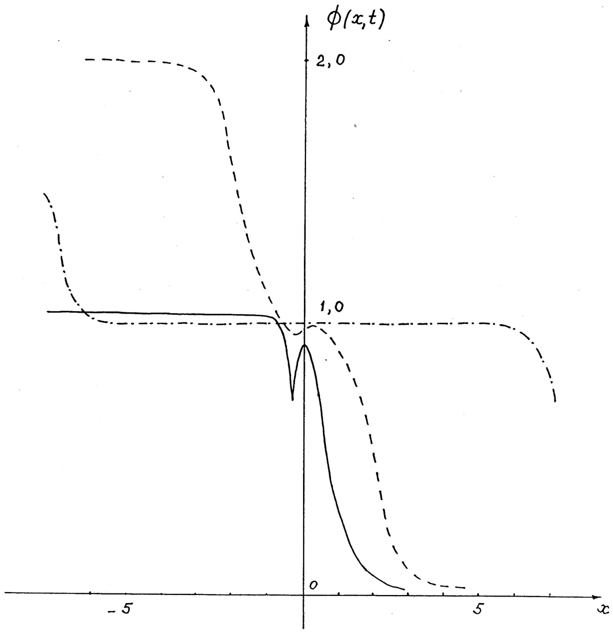

Fig. 2. The behaviour of $\phi(x,t)$ defined by Eq. (27) as a function of x for different t: $t = 0$ - solid line, $t = 3$ - dashed line and $t = 10$ dot-dashed line.

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NOVA RJEŠENJA U ϕ^4-TEORIJI S GUŠENJEM

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U okviru ϕ^4 teorije konstruirana su rješenja za dvojni prijelom i prijelom-antiprijelom s gušenjem.