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SOLITARY WAVES AND PERIODIC WAVES THROUGH AN ITERATIVE APPROACH

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We have obtained exact solutions of some non-linear partial differential equations by a new iteration method that was proposed by Blender. As examples, solitary wave solutions of the Benjamin-Bona-Mahony equation, Joseph-Egra equation, coupled K-dV equations and of the Zakharov-Kuznetsov equation have been obtained. For the K(2, 2) equation, recently suggested by Rosenau and Hyman (which give compactons: solitons of finite wavelength), a new type of periodic wave solutions has been found.

1. Introduction

Recently an iteration method has been suggested by Blender [1] to obtain exact solutions of some nonlinear partial differential equations. In contrast to the method of successive approximations of Picard [2], this method looks for the occurrence of the non-differentiated function (u) and solves for it. Boundary conditions, if any, are incorporated accordingly. One can use also some free parameters which can be adjusted at the end if they are not eliminated in the iteration procedure. The main point is that in some cases this method converges rapidly, typically after two steps to the exact solution. The convergence properties are not yet known. In this communication, we show that exact solutions of some non-linear partial differential equations such as the Benjamin-Bona-Mahony equation [3], Joseph-Egra equation [4], coupled K-dV equations, usually known as Hirota-Satsuma equations [5] and the Zakharov-Kuznetsov equation [6] can be extracted by such an iterative approach. For the K(2, 2) equation, recently suggested by Rosenau and Hyman [7], which give compactons, solitons of a finite wavelength, we report a new type of periodic wave solutions.

2. Formulation

We first proceed to obtain an exact solution of the Benjamin-Bona-Mahony equation [3], usually known as regularized long wave equation:

$$u_t + u_x + \alpha u u_x - u_{2xt} = 0. \tag{1}$$

According to the iterative method proposed by Blender [1], we define the iteration scheme as follows:

$$u^{n+1} = \frac{1}{\alpha}\left[\left(u^n_{2xt} - u^n_t - u^n_x\right)/u^n_x\right],\ n \geq 0. \tag{2}$$

With the clever choice of the wave ansatz, $u^0 = \sec a(x-ct)$, the solution converges within two steps to

$$u = \left(\frac{c-1}{\alpha}\right) - \frac{a^2 c}{\alpha}\left[8 + 12\tan^2 a(x-ct)\right]. \tag{3}$$

Strictly speaking, here the term *converges* is meant to say *converges to the form of the solution.* For example, here u converges to the function $\tan^2 a(x-ct)$, but not with the coefficient 12 and the first term of the square bracket equal to 8. This assertion is true for other cases we have dealt with here.

The boundary condition, $u \to 0$ as $|\, x \,| \to \infty$ gives $a = \mathrm{i}\sqrt{c - 1/c}/2$. Then Eq. (3) represents a one soliton solution

$$u = \frac{3(c-1)}{\alpha}\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{c-1}{c}}(x-ct)\right]. \tag{4}$$

We next consider Joseph-Egra equation [4], usually known as TRLW equation

$$u_t + u_x + \alpha u u_x + u_{x2t} = 0. \tag{5}$$

Here the iteration scheme is

$$u^{n+1} = \frac{1}{\alpha}\left[-\left(u^n_{x2t} + u^n_x + u^n_t\right)/u^n_x\right],\ n \geq 0. \tag{6}$$

With the wave ansatz $u^0 = \sec a(x-ct)$, the solution converges within two steps to

$$u = \left(\frac{c-1}{\alpha}\right) - \frac{a^2c^2}{\alpha}\left[8 + 12\tan^2 a(x-ct)\right]. \tag{7}$$

The boundary condition, $u \to 0$ as $\mid x \mid \to \infty$ gives $a = \mathrm{i}\sqrt{c-1}/(2c)$ and Eq. (7) then describes a one soliton solution

$$u = \frac{3(c-1)}{\alpha}\mathrm{sech}^2\left[\frac{\sqrt{c-1}}{2c}(x-ct)\right]. \tag{8}$$

We now consider the case of coupled K-dV equations, usually known as the Hirota-Satsuma equations [5]

$$u_t - a(u_{3x} + 6uu_x) = 2b\phi\phi_x \tag{9a}$$

$$\phi_t + \phi_{3x} + 3u\phi_x = 0. \tag{9b}$$

The iteration scheme, here, is

$$u^{n+1} = -\frac{1}{3}\left[\left(\phi_t^n + \phi_{3x}^n\right)/\phi_{3x}^n\right] \tag{10a}$$

$$\phi^{n+1} = \frac{1}{2b}\left[u_t^n\phi_x^n - au_{3x}^n\phi_x^n + 2a\left(\phi_t^n u_x^n + \phi_{3x}^n u_x^n\right)\right]/\left(\phi_x^n\right)^2\ ,\ n \geq 0. \tag{10b}$$

We take as the initial guess,

$$u^0 = \sec a_1(x-ct)$$

$$\phi^0 = b\sec a_1(x-ct)$$

and observe that it converges within two steps to

$$u = \frac{c}{3} - \frac{4a_1^2}{3}\left[2 + 3\tan^2 a_1(x-ct)\right] \tag{11a}$$

$$\phi = \frac{bc}{3}\left(2 + \frac{1}{a}\right) - \frac{4ba_1^2}{3}\left[2 + 3\tan^2 a_1(x-ct)\right]\ . \tag{11b}$$

The boundary condition, $u \to 0$ as $\mid x \mid \to \infty$ then gives

$$a_1 = \frac{\mathrm{i}\sqrt{c}}{2} \tag{12}$$

$$a = -1 \qquad . \tag{13}$$

Equations (11a,b) then take the form

$$u = c \cdot \mathrm{sech}^2 \left[\frac{\sqrt{c}}{2}(x - ct) \right] , \tag{14a}$$

$$\phi = bc \cdot \mathrm{sech}^2 \left[\frac{\sqrt{c}}{2}(x - ct) \right] . \tag{14b}$$

We now observe that u and ϕ will satisfy Eqs. (9a,b) if b is equal to $[-3/2(\sqrt{c} - 2)]^{1/3}$. Equations (14a,b) represent soliton solutions.

So far we have considered the equations in one space and in one time dimensions, usually termed as [1+1]D equations. We now consider equation of [2+1]D such as the Zakharov-Kuznetsov equation [6]

$$u_t + uu_x + u_{3x} + u_{x2y} = 0. \tag{15}$$

The iteration scheme for this equation is

$$u^{n+1} = \left[- \left(u_t^n + u_{3x}^n + u_{x2y}^n \right) / u_x^n \right] , \qquad n \geq 0. \tag{16}$$

With the wave ansatz

$$u^0 = \sec a(lx + my - ct),$$

the solution converges within two steps to

$$u = \frac{c}{l} - a^2(l^2 + m^2)[8 + 12 \tan^2 a(lx + my - ct)]. \tag{17}$$

The boundary condition, $u \to 0$ as $\mid x \mid \to \infty$ gives

$$a = \frac{\mathrm{i}}{2} \sqrt{\frac{c}{l(l^2 + m^2)}}.$$

Consequently, Eq. (17) takes the form of a solitary wave solution

$$u = \frac{3c}{l} \mathrm{sech}^2 \left[\sqrt{\frac{c}{4l(l^2 + m^2)}}(lx + my - ct) \right] . \tag{18}$$

Lastly we consider the K(2, 2) equation, recently suggested by Rosenau and Hyman [7] which appeared in the course of their study of the role of nonlinear dispersion in the formulation of nonlinear structures like liquid drops. Such an equation yields compacton as a particular solution, a soliton of a finite wavelength.

We try here to construct the exact solution of K(2,2) by the iterative method. The form of K(2,2) is

$$u_t + (u^2)_x + (u^2)_{3x} = 0$$

or

$$u_t + 2u(u_x + u_{3x}) + 6u_x u_{2x} = 0. \tag{19}$$

The iteration scheme would now be

$$u^{n+1} = -\frac{1}{2(u_x^n + u_{3x}^n)} \left[u_t^n + 6u_x^n u_{2x}^n \right]. \tag{20}$$

With the initial guess $u^0 = \cos 2a(x - vt)$, the solution converges within two steps to

$$u = -\frac{v}{2(4a^2 - 1)} + \left[\frac{24a^2}{2(4a^2 - 1)} \right]^2 \cos 2a(x - vt). \tag{21}$$

The condition, $u \to 0$ as $\mid x - vt \mid \to n\pi/a$, $(n = 0, \pm 1, \pm 2, ...)$ gives

$$v = \frac{(24a^2)^2}{2(4a^2 - 1)}. \tag{22}$$

Equation (21) then takes the form

$$u = -\left[\frac{v}{(4a^2 - 1)} \right] \sin^2 a(x - vt). \tag{23}$$

Here we observe that for u to satisfy Eq. (19), $\mid a^2 \mid$ should be equal to (1/16). Using then Eq. (22) we get $v = -3/2$. The exact solutions would then be

$$u = \pm 2 \sin^2 \left[\frac{1}{4}(x + \frac{3}{2}t) \right]. \tag{24}$$

Equation (24) represents special types of periodic waves (similar to the form of full wave rectified output) and not compactons, because here $u \to 0$ as $\mid x - vt \mid \to n\pi/a$, $(n = 0, \pm 1, \pm 2, ...)$. Such a periodic wave together with its anti-part (since amplitude can be $+2$ or -2) move in the negative direction of x-axis. Here we observe that the invariance of Eq. (19) under $u \to -u$ and $t = -t$ permits negative counterparts of such periodic waves moving in the positive direction of the x-axis.

3. Conclusion

In our computation, we have shown that a few exact solutions of some nonlinear partial differential equations can be extracted by a rather simple iterative process.

But the scheme limits itself where the terms u and its higher power together exist, for example in the equations $K(m,n) = u_t + (u^m)_x + (u^n)_{3x} = 0$, where $m = 2$, $n = 3$; $m = 3$, $n = 2$ and $m = 3$, $n = 3$. Nonlinear Klein-Gordon equation

$$u_{2t} - u_{2x} + \alpha u + \beta u^3 = 0$$

is another example. We also note further that such a scheme fails to give any solution for variable-coefficient nonlinear equations such as K-dV equation in a non-uniform medium

$$u_t + \gamma u + \alpha x u_x + 6uu_x + u_{3x} = 0,$$

generalized K-dV equation

$$u_t = u_{3x} + 6uu_x + [F(t)x + G(t)]\, u_x + 2F(t)u,$$

its modified version

$$u_t = u_{3x} - 6u^2 u_x + [F(t)x + G(t)]\, u_x + F(t)u$$

etc.

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SOLITONSKI VALOVI I PERIODIČKI VALOVI U ITERACIONOJ METODI

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Dobili smo egzaktna rješenja nekih nelinearnih parcijalnih diferencijalnih jednadžbi pomoću nove iteracione metode, predložene od Blendera. Kao primjer konstruirana su solitonska rješenja Benjamin-Bona-Mahonyeve jednadžbe, Joseph-Egrine jednadžbe, vezanih K-dV jednadžbi, te Zakharov-Kuznetsovljeve jednadžbe. Za K(2,2) jednadžbu, nedavno predloženu od strane Rosenaua i Hymana (koja daje kompaktone: solitne konačne valne duljine), pronađen je novi tip periodičkih valnih rješenja.