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ON TWO – PARAMETER DEFORMATIONS OF SU(1,1) ALGEBRA AND ASSOCIATED SPIN CHAINS

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Using Fadeev-Reshetikhin-Takhtajan procedure, we analyse conditions under which two "quantum" SU(1,1) superalgebras are isomorphic as Hopf algebras. In the light of this results we discuss spin chains invariant under multiparameter SU(1,1).

1. *Introduction*

During the past few years much work has been done to clarify various aspects of "quantum" deformations of the simple Lie algebras and superalgebras [1]. Attention has been focussed mostly on the one-parameter deformations of SU(2) and SU(1,1) algebras because they are recognized as underlying symmetries of some interesting physical models, e.g. asymmetric Heisenberg-like spin chains [2], WZW and Chern--Simons field theories [3], etc.

Recently, multiparameter deformations of Lie (super) algebras were also proposed [4]; however some confusion appeared concerning physical relevance of additional parameters. We mention recent examples. The two-parameter $\mathrm{SU}_{q,\eta}(2)$ was considered in Ref. 5 and it was shown that the η parameter could be removed from the algebra of generators, but it still appeared in the coproduct and the antipode, i.e. in the coalgebra structure[1]. Hence, addition of the angular momentum or the $\mathrm{SU}_{q,\eta}(2)$ ($\mathrm{SU}_{q,\eta}(1,1)$) invariant interaction in spin chains would depend on the two parameters (q,η) in a non-trivial way. The Clebsch-Gordan coefficients for $\mathrm{SU}_{q,\eta}(2)$ were calculated in Ref. 7 and the two-parameter dependence of the C.G. coefficients was reported.

In Ref. 8 a particular XY spin $-1/2$ chain with the nearest–neighbour interaction was used to define a new two-parameter $\mathrm{SU}_{q,\eta}(1,1)$. It was claimed that a non-trivial two parameter algebra and a coalgebra structure were found.

In Ref. 9 we worked out in detail $\mathrm{SU}_{q,\eta}(2)$ [5], showing that it is isomorphic to $\mathrm{SU}_q(2)$ (in the sense of Hopf algebra). The appearance of the parameter η is artificial and of no physical relevance. Therefore, the Clebsch-Gordan coefficients for $\mathrm{SU}_{q,\eta}(2)$ are essentially one parametric, in contrast to the calculation of Ref. 7.

In this paper we extend our analysis [9] to the $\mathrm{SU}_{q,\eta}(1,1)$ of Ref. 6 and draw a similar conclusion. We demonstrate that $\mathrm{SU}_{q,\eta}(1,1)$ and $\mathrm{SU}_q(1,1)$ are isomorphic and the η parameter drops out of the coalgebra and the algebra. In the light of these results we discuss the SU(1,1) deformation constructed by Hinrichsen and Rittenberg [8].

2. *Isomorphism between $SU_{q,\eta}(1,1)$ and $SU_q(1,1)$*

In this section we prove that two-parameter $\mathrm{SU}_{q,\eta}(1,1)$ of Ref. 6 and $\mathrm{SU}_{q,\eta=1}(1,1)$ are isomorphic as Hopf algebras.

We use Fadeev-Reshetikhin-Takhtajan (FRT) "quantization" procedure [10]:

$$R_{q,\eta}\, P\, L_1(\varepsilon)\, P\, L_1(\varepsilon') = P\, L_1(\varepsilon')\, P\, L_1(\varepsilon)\, R_{q,\eta}$$
$$(\varepsilon,\varepsilon') = (+,+);(-,-);(+,-)$$

$$R_{q,\eta} = \begin{pmatrix} q & & & \\ & q-q^{-1} & \eta^{-1} & \\ & \eta & 0 & \\ & & & -q^{-1} \end{pmatrix} \qquad P = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & -1 \end{pmatrix} \tag{2.1}$$

$$L_1(\varepsilon) = L(\varepsilon)\otimes 1$$
$$L(+) = \begin{pmatrix} L^+_{11} & L^+_{12} \\ 0 & L^+_{22} \end{pmatrix} \qquad L(-) = \begin{pmatrix} L^-_{11} & 0 \\ L^-_{21} & L^-_{22} \end{pmatrix}$$

[1] The same was observed for the $\mathrm{SU}_{q,\eta}(1,1)$ case [6]

to obtain the $SU_{q,\eta}(1,1)$ algebra as follows[2]:

$$
\begin{aligned}
L_{12}^{+}L_{ii}^{+} &= q\eta\, L_{ii}^{+}L_{12}^{+} \qquad & L_{12}^{+}L_{ii}^{-} &= q^{-1}\eta\, L_{ii}^{-}L_{12}^{+} \\
L_{21}^{-}L_{ii}^{-} &= q\eta^{-1}\, L_{ii}^{-}L_{21}^{-} \qquad & L_{21}^{-}L_{ii}^{-} &= q\eta^{-1}\, L_{ii}^{+}L_{21}^{-}
\end{aligned}
$$

$$
\eta^{-1}\, L_{12}^{+}L_{21}^{-} - \eta\, L_{21}^{-}L_{12}^{+} = (q-q^{-1})\,(L_{11}^{+}L_{22}^{-} - L_{22}^{+}L_{11}^{-}) \tag{2.2}
$$

$$
\begin{aligned}
[L_{ii}^{\pm}, L_{jj}^{\pm}] &= [L_{ii}^{+}, L_{jj}^{-}] = 0 \\
(L_{12}^{+})^2 &= (L_{21}^{-})^2 = 0\,.
\end{aligned}
$$

$SU_{q,\eta}(1,1)$ is endowed with a coalgebra (super – Hopf) structure if the coproduct Δ, the antipode S and the counit ε are defined (correct Z_2 grading should be taken into account)

$$
\begin{aligned}
\Delta(L_{ij}^{\pm}) &= \textstyle\sum_k L_{ik}^{\pm} \otimes L_{kj}^{\pm} \\
S(L)L &= 1 \\
\varepsilon(L) &= 1.
\end{aligned} \tag{2.3}
$$

The well known one-parameter SU(1,1) algebra is reproduced by setting $\eta = 1$ in Eqs. (2.1) and (2.2).

The following proposition holds:

Proposition 1. *Two "quantum" SU(1,1) (super)algebras are isomorphic as Hopf algebras if their R matrices are related by a similarity (gauge) transformation of diagonal form. Particularly, the algebras $SU_{q,\eta}(1,1)$ and $SU_{q,\eta=1}(1,1)$ are isomorphic since*

$$
\begin{aligned}
R_{q,\eta} &= V(\eta) R_{q,\eta=1} V(\eta)^{-1} \\
V(\eta) &= 1 \otimes \eta^{J_0} \\
J_0 &= 1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned} \tag{2.4}
$$

It is easy to extend this proposition to the multiparameter case with the diagonal gauge – transformation V depending on several parameters.

The proof is simple. Substitution of (2.4) into (2.1) yields

$$
\begin{aligned}
R_{q,\eta=1} P\mathcal{L}_1(\varepsilon) P\mathcal{L}_1(\varepsilon') &= P\mathcal{L}_1(\varepsilon') P\mathcal{L}_1(\varepsilon) R_{q,\eta=1} \\
\mathcal{L}_1(\varepsilon) &= (\eta^{-J_0} \otimes 1) L_1(\varepsilon)(1 \otimes \eta^{J_0}) \\
\mathcal{L}_2(\varepsilon) &= P\mathcal{L}_1(\varepsilon) P\,.
\end{aligned} \tag{2.5}
$$

[2] This is the same algebra as that used in e.g. Dabrowski and Wang paper [6], after identification $q \to q/\eta$ and $p \to q\eta$.

The algebra that emerges from (2.5) is the one-parametric SU(1,1). Notice that the gauge transformation preserves the triangular form of $\mathcal{L}(\varepsilon)$ changing only the definitions of the generators. The relations between the generators of $\mathrm{SU}_{q,\eta}(1,1)$ (L) and $\mathrm{SU}_{q,\eta=1}(1,1)$ ($\mathcal{L}$) are

$$\begin{aligned}
\mathcal{L}^+_{11} &= L^+_{11}\eta^{J_0-1/2} \qquad & \mathcal{L}^-_{11} &= L^-_{11}\eta^{J_0-1/2} \\
\mathcal{L}^+_{12} &= L^+_{12}\eta^{J_0-1/2} \qquad & \mathcal{L}^-_{21} &= L^-_{21}\eta^{J_0+1/2} \\
\mathcal{L}^+_{22} &= L^+_{22}\eta^{J_0+1/2} \qquad & \mathcal{L}^-_{22} &= L^-_{22}\eta^{J_0+1/2} \, .
\end{aligned} \tag{2.6}$$

The coalgebra structure consistent with (2.5) is defined as in (2.3) with L replaced by $\mathcal{L}$. Hence, no additional parameter appears in the coproduct and the antipode, which was not recognized in Ref. 6. However, one can relate the two structures and, by virtue of (2.6) obtains e.g.

$$\begin{aligned}
\Delta(\mathcal{L}^+_{1j}) &= \Delta(L^+_{1j})\Delta(\eta^{J_0-1/2}) \\
S(\mathcal{L}^+_{1j}) &= S(\eta^{J_0-1/2})S(L^+_{1j}) \\
\varepsilon(\mathcal{L}^+_{1j}) &= \varepsilon(\mathcal{L}^+_{1j})\varepsilon(\eta^{J_0-1/2}) \, .
\end{aligned} \tag{2.7}$$

We also state the inverse of Proposition 1, namely

Proposition 2. *If the two "quantum" SU(1,1) (super)algebras are isomorphic, their R matrices are related by a similarity transformation V of diagonal form.*

The proof follows immediately after inserting $V V^{-1} = 1$ into FRT equations. That V has to be a diagonal matrix follows from the triangular structure of L and $\mathcal{L}$.

The results may be concisely displayed in the diagram (Fig.1). It is a matter of convenience which path in the diagram is preferred to use. Physically, the paths are equivalent.

Two remarks are in order. First, the same gauge transformation $V(\eta)$ connects the $R_{q,\eta}$ and $R_{q,\eta=1}$ matrices of $\mathrm{SU}_{q,\eta}(2)$ and $\mathrm{SU}_{q,\eta=1}(2)$, respectively. Second, 4×4 constant R matrices of the eight-or-less vertex form (2.1) are known [11] and none of them generates the non-trivial two-parameter SU(1,1) superalgebra.

3. $SU_{q,\eta}$(1,1) and spin – chain Hamiltonians

We briefly discuss a particular construction of spin chains invariant under some "quantum" algebra $\mathcal{A}$ [12].

It is essential for the FRT procedure that the R matrix should satisfy the Artin braid – group relations

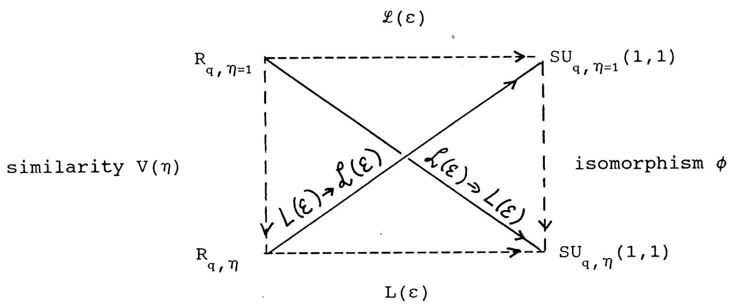

Fig. 1. Isomorphism of $SU_{q,\eta}$(1,1) and $SU_{q,\eta=1}$(1,1).

$$
\begin{aligned}
&R_k R_{k+1} R_k = R_{k+1} R_k R_{k+1} \\
&[R_k, R_{k'}] = 0 \quad \forall\, k-k' > 1 \\
&R_k = 1 \otimes \ldots \otimes R \otimes \ldots \otimes 1\,.
\end{aligned} \tag{3.1}
$$

Consistency with the "quantum" algebra $\mathcal{A}$ requires that R commutes with the coproduct [1]

$$[R, \Delta(\mathcal{A})] = 0\,. \tag{3.2}$$

Identification of R_k with the Hamiltonian density H_k, which represents the two-body interaction between $(k, k+1)$ sites on the chain

$$R_k = H_k + \omega 1 \quad \omega = \text{ const.} \tag{3.3}$$

gives a spin chain with the total Hamiltonian $H = \sum H_k$. Owing to (3.2) H commutes with the action of $\mathcal{A}$.

When $\mathcal{A} \equiv \mathrm{SU}_q(1,1)$, a supersymmetric generalization of Lai-Sutherland spin chains [13] is reproduced. The role of the permutation operator is played by the R matrix (2.1) with $\eta = 1$.

When $\mathcal{A} \equiv \mathrm{SU}_{q,\eta}(1,1)$ and $\eta \neq 1$, the Hamiltonian (3.3) is generally not Hermitian. We distinguish two cases:

(i) $R_{q,\eta}$ is the non-hermitian matrix and (q,η) are real parameters. The Hamiltonian $H(q,\eta)$ built from $R_{q,\eta}$ is also non-Hermitian and depends on two parameters (q,η). It can be related to the one-parameter Hermitian Hamiltonian $H(q,\eta=1)$ by the non-unitary gauge transformation $V(\eta) = 1 \otimes \eta^{J_0}$. $H(q,\eta)$ and $H(q,\eta=1)$, being similar (with the same set of eigenvalues), should have exactly the same thermodynamic properties [14].

(Remark: If we define the Hamiltonian as $\tilde{H}(q,\eta) = (R_{q,\eta} + R^{+}_{q,\eta})/2 = (R_{q,\eta} + R_{q,\eta^{-1}})/2$, it is obviously Hermitian and depends on two parameters (q,η), but the linear combination $(R_{q,\eta} + R_{q,\eta^{-1}})$ does not solve the braid condition (3.1). The FRT procedure cannot be applied in this case. $\tilde{H}(q,\eta)$ has a different set of eigenvalues than $H(q,\eta)$ and cannot be obtained from the $\eta = 1$ case by gauge transformation.)

(ii) $R_{q,\eta}$ is the hermitian matrix if q is real and $\eta = \mathrm{e}^{\mathrm{i}\Phi}$ is a complex parameter. The Hamiltonian $H(q,\eta)$ is Hermitian and depends on two parameters (q,η). The unitary gauge transformation $V(\eta) = (1 \otimes \eta^{J_0}) = V(\eta^{-1}) = (V(\eta)^{+})^{-1}$ transforms it to the one-parameter $\mathrm{SU}_q(1,1)$ – invariant Hamiltonian $H(q,\eta=1)$ of the Lai-Sutherland type.

We notice that same discussion applies to the $\mathrm{SU}_{q,\eta}(2)$ case as the transformation $V(\eta)$ also connects one- and two-parameter R matrices. Particularly, this means that there are no $\mathrm{SU}_{q,\eta}(2)$ – invariant Hermitian Hamiltonian for (q,η) real parameters.

In the light of these results we would like to comment on Hinrichsen and Rittenberg's realization of "$\mathrm{SU}_{q,\eta}(1,1)$" [8].

They claimed that the Hamiltonian

$$H(q,\eta) = \begin{pmatrix} q+q^{1} & 0 & 0 & \eta-\eta^{-1} \\ 0 & q-q^{-1} & \eta+\eta^{-1} & 0 \\ 0 & \eta+\eta^{-1} & -(q-q^{-1}) & 0 \\ \eta-\eta^{-1} & 0 & 0 & -(q+q^{-1}) \end{pmatrix} = H(q,\eta)^{+} \tag{3.4}$$

respected the particular "$\mathrm{SU}_{q,\eta}(1,1)$" algebra, realized by Cartesian generators T_x, T_y and E. (For the definition of the algebra, we refer to their paper.) Several objections to this constructions may be raised immediately. It appears that their "$\mathrm{SU}_{q,\eta}(1,1)$" has only three generators. The usual definition of SU(1,1) also includes the fourth, diagonal generator $J_0 \equiv S_z$ with the coproduct $\Delta(S_z) = S_z \otimes 1 + 1 \otimes S_z$ and the non-trivial commutators with T_x, T_y. It is obvious that the commutator $[H(q,\eta), \Delta(S_z)]$ is not zero. (We do not agree with their relation (30). Instead we would obtain $R\Delta = (\Delta R)^T$). In the limit $q,\eta \to 1$, "$\mathrm{SU}_{q,\eta}(1,1)$" does not reduce to SU(1,1) with four generators unless S_z is put by hand. The R matrix associated to $H(q,\eta)$ as

$$R_{q,\eta} = H(q,\eta) + \frac{1}{2}\sqrt{(q-q^{-1})^2 + (\eta-\eta^{-1})^2} \tag{3.5}$$

does not satisfy the Artin braid (3.1) and "$\mathrm{SU}_{q,\eta}(1,1)$" is not obtainable from the super-FRT procedure (2.1). Hence, we suspect that it is really a two-parameter SU(1,1) super – Hopf algebra.

4. Conclusion

We briefly summarize the main results of this paper.

Using the Fadeev-Reshetikhin-Takhtajan "quantization" procedure, we have defined multiparameter and one-parameter SU(1,1) superalgebras. By noticing that their R matrices are connected by similarity transformation, represented by a diagonal matrix, we have proved that multiparameter SU(1,1) algebras are isomorphic to one parameter SU(1,1) as Hopf algebras. Inversely, if two algebras are isomorphic, their R matrices are similar.

In the light of these results we consider $\mathrm{SU}_{q,\eta}(1,1)$ invariant spin chains. We conclude that spin – chain Hamiltonian, invariant under multiparameter SU(1,1) algebra (obtainable from the FRT procedure), can always be transformed to Hamiltonian invariant under the one-parameter SU(1,1). This is achieved by similarity transformation of diagonal form.

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O DVOPARAMETARSKIM DEFORMACIJAMA ALGEBRE SU(1,1) I SPINSKIM LANCIMA

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Pomoću Fadeev-Reshetikhin-Takhtajan procedure analizirani su uvjeti pod kojima su dvije "kvantne" SU(1,1) algebre izomorfne kao Hopfove algebre. Na osnovi tih rezultata raspravljeni su spinski lanci invarijantni na multiparametarske SU(1,1) algebre.