Printed ISSN 1330–0016
Online ISSN 1333–9133
CD ISSN 1333–8390
CODEN FIZBE7

LETTER TO THE EDITOR

LYRA COSMOLOGY IN $D = n + 4$ DIMENSIONS: THE INTERESTING CASES OF $n \geq 6$ AND $n \leq 104$

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Received 28 March 2007; Revised manuscript received 25 February 2009
Accepted 17 June 2009 Online 13 July 2009

Exact solutions of the Einstein's field equations in Lyra geometry in $D = 4 + n$ dimensional spacetime with time-increasing G and decaying cosmic fluid and density are obtained and discussed in some details. We suggest the phenomenological dissipative law $\beta^2 \propto H/t$ for the square of the non-zero component of the displacement vector field ϕ_A. H is the Hubble parameter in the extra-dimensional spacetime metric. It has been observed that the cases $n \geq 6$ are within the observational limit, the four-dimensional spacetime is in the stage of accelerated expansion, while the extra-dimensions are contracting in time.

PACS numbers: 04.20.Jb; 98.80.Hw UDC 52-423, 524.83

Keywords: higher-dimensional cosmology, gravitational constant, Lyra geometry, effective pressure

It is very interesting that many of the new developments and achievements in theoretical high-energy physics required the introduction of extra-dimensions. We believe today that extra-dimensions play a crucial role at the high energy as at the cosmological level. In fact, they are considered as one of the most interesting attempts to solve some of the major problems in fundamental physics, including the initial inevitable singularity in the cosmological past and the hierarchy problem in particle physics [1−5]. There exist in literature many extra-dimensional theories following the theoretical progress in mathematical physics, including the modern Kaluza-Klein (KK) theories, where the isometries of the extra dimensions are gauge symmetries, supersymmetry as a powerful symmetry theory combining fermions and bosons, supergravity which are field theories having the local supersymmetry (the spacetime is enlarged by adding fermionic coordinates), supergravities in various spacetime dimensions and their relevance to string dualities, superstrings (string theory with fermions), M-theory, which is supposed to be the eleven-dimensional theory underlying the five known superstring theories and eleven-dimensional supergravity.

In return to cosmologies governed by the Einstein's field equations, it has been observed that extra-dimensions may provide a possible solution to the flatness and horizon problem by producing large amount of entropy during contraction process, as compared to the standard inflationary scenario [6, 7]. Others claimed that time-variation of fundamental constants in physics could produce the evidence of extra-dimensions [8]. In the last few decades, there has been much interest to study higher-dimensional cosmological models within the context of Lyra geometry [9 − 13]. Lyra geometry, based on Einstein's idea of geometrizing gravitational field in the form of general relativity theory, is of interest because it produces effects similar to those produced in Einstein's field theory. In particular, the vector field in the theory plays a similar role as the Einstein's cosmological constant. Most of these studies were done in five-dimensional spacetime geometry of the universe in which gravitational constant varies with time. In this work, we will explore Lyra geometry in $D = 4+n$ dimensions. Moreover, we will assume that the square of the non-zero component of the displacement vector field ϕ_A decays as α/H where α is a positive parameter and $H(t)$ is the Hubble parameter in the higher-dimensional metric.

In Lyra geometry, the Einstein's field equations in $D = 4 + n$ dimensions read as

$$R_{AB} - \frac{1}{2}g_{AB}R + \frac{3}{2}\phi_A\phi_B - \frac{3}{4}g_{AB}\phi_C\phi^C = -\kappa T_{AB}\,. \tag{1}$$

The displacement vector is defined as $\phi_A = (\beta, 0, 0, \ldots, 0)$ and the energy-momentum tensor is defined usually as $T_{AB} = (p + \rho)u_A u_B - p g_{AB}$ for $A, B = 1, 2, 3, \ldots, 4+n$, and $p = p_a + p_b$ is the total pressure of the cosmic fluid and ρ is its corresponding density We consider in the following a spatially flat Friedmann-Robertson-Walker (FRW) metric extended with two scale factors [14]

$$\mathrm{d}s^2 = g_{AB}\,\mathrm{d}X^A\mathrm{d}X^B = \mathrm{d}t^2 - a^2(t)\,\gamma_{ij}\,\mathrm{d}x^i\mathrm{d}x^j + b^2(t)\,\widetilde{\gamma}_{kl}\,\mathrm{d}y^k\mathrm{d}y^l\,. \tag{2}$$

Here i, $j = 1$, 2, 3 and k, $l = 4$. The field equations look like:

$$3\left(\frac{\dot{a}^2}{a^2} + n\,\frac{\dot{a}}{a}\,\frac{\dot{b}}{b}\right) + \frac{n(n-1)}{2}\,\frac{\dot{b}^2}{b^2} = \kappa\,\rho + \frac{3}{4}\beta^2\,, \tag{3}$$

$$2\,\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 2n\,\frac{\dot{a}}{a}\,\frac{\dot{b}}{b} + n\,\frac{\ddot{b}}{b} + \frac{n(n-1)}{2}\,\frac{\dot{b}^2}{b^2} = -\kappa\,p_a - \frac{3}{4}\beta^2\,, \tag{4}$$

$$3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + 3(n-1)\,\frac{\dot{a}}{a}\,\frac{\dot{b}}{b} + (n-1)\,\frac{\ddot{b}}{b} + \frac{(n-1)(n-2)}{2}\,\frac{\dot{b}^2}{b^2} = -\kappa\,p_b - \frac{3}{4}\beta^2\,. \tag{5}$$

It is obvious that the term $3\beta^2/4$ plays the role of the cosmological constant Λ. For convenience, we will assume that $p_a = p_b + \xi$, where ξ is a pressure-like quantity playing the role of the bulk viscosity. Thus, subtracting Eq. (5) from (4) gives easily

$$\frac{\ddot{a}}{a} + 2\,\frac{\dot{a}^2}{a^2} + (n-3)\,\frac{\dot{a}}{a}\,\frac{\dot{b}}{b} - \frac{\ddot{b}}{b} + (n-1)\,\frac{\dot{b}^2}{b^2} = \kappa\,\xi\,. \tag{6}$$

In what follows, we will assume a power-law form of the scale factors $a(t)$ and $b(t)$ of the type $a(t) \propto t^p$ and $b(t) \propto t^q$, where p and q are real parameters. Further, we will assume that the gravitational constant, the extra-pressure ξ and the cosmic fluid density and pressure vary with time like $G(t) = G_0(t/t_0)^r$, $\xi(t) = \xi_0(t_0/t)^s$, $\rho(t) = \rho_0(t_0/t)^x$, $p_b(t) = p_{0,b}(t/t_0)^y$, and $p_a(t) = p_{0,a}(t/t_0)^z$, r, s, x, y and z are real parameters, while G_0, ξ_0, ρ_0, $p_{0,b}$ and $p_{0,a}$ are the present values of G, ξ, ρ, p_b and p_a. Equation (6) consequently gives[1]

$$\frac{3p^2 - p + (n-3)pq - 2q^2 + q + nq^2}{t^2} = 8\pi G_0 \xi_0 \left(\frac{t}{t_0}\right)^{r-s}. \tag{7}$$

Thus, a consistent relation corresponds to $r = s - 2$. Note that for $s = 2$, $r = 0$, i.e. G is constant with time. For $s > 2$, $r > 0$, and hence G increases in time, while the cosmic fluid density decreases in time.

We will assume in what follows the phenomenological law

$$\beta^2 = \alpha \frac{t_0}{t}\frac{H}{H_0} = \alpha \frac{t_0}{t}\frac{1}{H_0}\left(\frac{\dot{a}}{a} + \frac{1}{3}\frac{\dot{b}}{b}\right) = \alpha \frac{t_0}{t^2}\frac{1}{H_0}\left(p + \frac{1}{3}q\right), \tag{8}$$

where $H = (\dot{a}/a) + (1/3)(\dot{b}/b) = (3p+\rho)/(3t)$, $\alpha = \beta_0^2$, the present value of β and H the present value of H_0. Consequently, Eqs. (3)$-$(5) give:

$$8\pi G_0 \rho_0 \left(\frac{t}{t_0}\right)^{r-x} + \frac{3}{4}\frac{t_0}{t^2}\frac{\alpha}{H_0}\left(p + \frac{1}{3}q\right) = \frac{1}{t^2}\left(3p^2 + 3npq + \frac{n(n-1)}{2}q^2\right), \tag{9}$$

$$\begin{aligned} &-8\pi G_0 p_{0,b}\left(\frac{t}{t_0}\right)^{r-z} - 8\pi G_0 \xi_0 \left(\frac{t_0}{t}\right)^2 - \frac{3}{4}\frac{t_0}{t^2}\frac{\alpha}{H_0}\left(p + \frac{1}{3}q\right) \\ &= \frac{1}{t^2}\left(3p^2 - 2p + 2npq + \frac{n(n-1)q^2 + 2nq(q-1)}{2}\right), \end{aligned} \tag{10}$$

$$\begin{aligned} &-8\pi G_0 p_{0,b}\left(\frac{t}{t_0}\right)^{r-z} - \frac{3}{4}\frac{t_0}{t^2}\frac{\alpha}{H_0}\left(p + \frac{1}{3}q\right) \\ &= \frac{1}{t^2}\left(6p^2 - 3p + 3(n-1)pq + (n-1)q(q-1) + \frac{(n-1)(n-2)}{2}q^2\right), \end{aligned} \tag{11}$$

and thus from Eqs. ($-$(11), a compatible equation corresponds to $x - r = 2$, $z - r = 2$ with $x = y = z = s$. Equations (9) and (10) are rewritten like:

$$8\pi G_0 \rho_0 t_0^2 + \frac{3t_0}{4}\frac{\alpha}{H_0}\left(p + \frac{1}{3}q\right) = 3p^2 + 3npq + \frac{n(n-1)}{2}q^2, \tag{12}$$

[1]The reader may be refereed to the recent works of Demianski et al. and Amendola et al. [15] to explore the important role of scaling solutions in accelerating cosmology.

$$-8\pi G_0 p_{0,b} t_0^2 - 8\pi G_0 \xi_0 t_0^2 - \frac{3t_0}{4}\frac{\alpha}{H_0}\left(p + \frac{1}{3}\,q\right)$$
$$= 3p^2 - 2p + 2npq + \frac{n(n-1)q^2 + 2nq(q-1)}{2}\,, \tag{13}$$

$$-8\pi G_0 p_{0,b} t_0^2 - \frac{3t_0}{4}\frac{\alpha}{H_0}\left(p + \frac{1}{3}\,q\right)$$
$$= 6p^2 - 3p + 3(n-1)pq + (n-1)q(q-1) + \frac{(n-1)(n-2)}{2}q^2\,. \tag{14}$$

In fact, the parameter t_0 may be approximated then by $t_0 \sim \sqrt{3/(8\pi G_0 \rho_0)}$ which results from the standard Friedmann equations in the inflationary paradigm [16]. Subtracting Eq. (14) from (13) gives

$$3 = 3p^2 - p + (n-3)pq + q - nq^2\,. \tag{15}$$

An interesting feature may arise if we take $n = 3$, thus $3 = (p-q)(3p+3q-1)$ and at least two possible solutions may exist: $p-q = 1$ and $3p+3q = 4$, yielding $q = 1/6$ and $p = 19/6$ or $p - q = 3$, and $3p + 3q = 2$, yielding $q = -7/6$ and $p = 11/6$. The second case is of interest because it corresponds to a seven-dimensional spacetime where the FRW scale factor accelerates with time while the extra-dimensional scale factor decreases with time as a power law. In the first case $Ht \approx 6.55$, which is ruled out by the observational limits, while the second case gives $Ht \approx 1.44$ and this is an acceptable age of the universe. Admitting the second solution, Eq. (12) gives $\alpha \approx -6.9H_0/t_0 < 0$ and thus $\beta^2 < 0$ as is expected. It has been pointed out that $\beta^2 \approx -4\Lambda/3$ and thus $\Lambda > 0$. In the case $n = 1$ (five-dimensional spacetime), $3 = (p-q)(3p+q-1)$ and again at least two possible solutions may exist $p-q = 1$ and $3p + q = 4$, yielding $p = 5/4$ and $q = 1/4$. The fifth dimension in this special case increases with time. The second possible case corresponds to $p - q = 3$ and $3p + q = 2$ giving $p = 5/4$ and $q = -7/4$. This particular case is appealing since it gives an accelerating universe and a contracting fifth dimension. In association with the case $n = 3$, the seven-dimension case is more interesting because a more accelerating expansion occurs. Further, for $n = 6$, $3 = (p-q)(3p + 6q - 1)$.

At least two possible solutions may exist: the first corresponds to $p-q = 3$ and $3p + 6q = 2$ for which $p = 20/9$ and $q = -7/9$, and the second case corresponds to $p - q = 1$ and $3p + 6q = 4$ for which $p = 10/9$ and $q = 1/9$. In fact, it may be proved that for all $n \geq 6$, at least one of the acceptable solutions corresponds to an accelerated expansion, while the extra-dimensional scale factor decreases with time. All the previous cases hold even in the presence of dissipation or bulk viscosity $\Pi\,(\propto \xi)$, for which the effective pressure turns out to be $p_{\text{effective}} = p + \Pi$. This indicates that for n-dimensional spacetime, the extra dimension is either expanding at a very slow rate or collapsing, while the three others (for $D = 5$) or the five others (for $D = 7$) continue to expand.

More generally, assuming the relation $p = Nq$ or $a(t) \approx b^N(t)$. From Eq. (15), we obtain easily the quadratic equation

$$\big(3N^2 + (n-3)N - n\big)q^2 + (1-N)q - 3 = 0\,, \tag{16}$$

and two values of N may exist if we admit the unique solution

$$q = \frac{37(n+3)}{n^2 - 105n + 9}\,, \tag{17}$$

for which the discriminant is equal to zero and thus $N_1 = 1$ and $N_2 = (1-12n)/37$. Thus the value of N_2 is more interesting. For $n = 1$, $N_2 \approx -0.3$, while for $n = 3$, $N_2 \approx -0.94$. Thus for $n = 1$, $q = -1.55$ and $p = 0.465$, while for $n = 3$, $q = -0.75$ and $p = 0.705$. Thus a is expanding and b is contracting. It may be easily proved that for $n = 6$, $N_2 = -1.918$, $q = -0.57$ and $p = 1.1$. For $n = 7$, $N_2 = -2.24$, $q = -0.54$ and $p = 1/2$. For $n = 10$, $N_2 = -3.21$, $q = -0.511$ and $p = 1.64$, in agreement with the recent astrophysical observations [17−19]. Cosmology in ten dimensions was proven to hold important characteristics and features [20, 21]. It is worth noting that as long as $n \leq 104$, $p > 0$ and $q < 0$. For a much higher dimension, q approaches zero (constant extra-dimension) and $p \to -12$ (contracting to four-dimensional spacetime). Cosmology in six dimensions and higher is explored in the literature and many important features were revealed [22−26].

In summary, in the present short communication we have investigated the $D = 4 + n$ cosmological model with dynamical gravitational constant and cosmic fluid density. Moreover, we have assumed that $p_a = p_b + \xi$, where ξ is a pressure-like quantity playing the role of the bulk viscosity. We have assumed power-law behavior for G, ρ, p_a, p_b and ξ, and furthermore we suggested the dissipative law for the square of the non-zero component of the displacement vector field ϕ_A, $\beta^2 \propto \alpha H/t$. Exact solutions of the Einstein's field equations are obtained. It has been observed that the gravitational constant increases in time, while ρ, p_a, p_b and ξ decrease in time. Assuming the relation $p = Nq$ or $a(t) \approx b^N(t)$, we noticed that as long as $n \geq 6$, a is expanding and b is contracting. The case $n = 10$ (eleven-dimensional spacetime) fits well with supernova and radio sources observations. While a vast literature exists to address the observational fact of the current expansion and evolution of the universe from the higher-dimensional point of view, we are not aware of models similar to the one developed in this paper. Further details, consequences and numerical confrontations with observations are in progress.

Acknowledgements

The author would like to thank the anonymous referees for their useful comments and precious suggestions.

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LYRINA KOZMOLOGIJA U $D = n + 4$ DIMENZIJA: ZANIMLJIVI SLUČAJEVI $n \geq 6$ I $n \leq 104$

Postigli smo i podrobno raspravljamo egzaktna rješenja Einsteinovih jednadžbi polja u Lyrinoj geometriji i $D = 4 + n$ dimenzijskom prostoru-vremenu, s vremenski rastućim G i opadajućom kozmičkom tekućinom i gustoćom. Predlažemo fenomenološki zakon disipacije $\beta^2 \propto H/t$ za kvadrat ne-nulte sastavnice vektora posmačnog polja ϕ_A. H je Hubbleov parametar u dodatno-dimenzijskoj prostorno-vremenskoj metrici. Nalazi se da su slučajevi $n \geq 6$ u granicama opažanja, a četiri-dimenzijski prostor-vrijeme se ubrzano širi dok se dodatne dimenzije sužavaju s vremenom.