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LETTER TO THE EDITOR

POLYGONAL DERIVATION OF THE NEUTRINO MASS MATRIX

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Dedicated to the memory of Professor Dubravko Tadić

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Representations of the symmetry group D_n of the $n-$sided regular polygon have generic multiplication rules if n is prime. Using D_n with $n = 5$ or greater, a particular well-known form of the Majorana neutrino mass matrix is derived.

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The form of the 3×3 Majorana neutrino mass matrix $\mathcal{M}_\nu$ has been the topic of theoretical study for some time. If $\mathcal{M}_\nu$ has less than the full 6 parameters, then there exists at least one relationship among masses and mixing angles, which may be tested against the increasingly more precise experimental data from neutrino oscillations. However, even if such a comparison is successful, the question still remains as to why it has such a form. A possible answer is that it comes from an underlying symmetry. In this paper, it is shown how

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix} \tag{1}$$

may be derived from D_n, the symmetry group of the regular $n-$sided polygon, where n is a prime number, equal to or greater than 5.

Consider D_5, the symmetry group of the regular pentagon. It has 10 elements, 4 equivalence classes, and 4 irreducible representations. Its character table is given in Table 1.

TABLE. 1. Character table of D_5.

class	n	h	χ_1	χ_2	χ_3	χ_4
C_1	1	1	1	1	2	2
C_2	5	2	1	-1	0	0
C_3	2	5	1	1	$\phi-1$	$-\phi$
C_4	2	5	1	1	$-\phi$	$\phi-1$

Here n is the number of elements and h is the order of each element. The number ϕ is the Golden Ratio (or Divine Proportion) known to the ancient Greeks

$$\phi = \frac{\sqrt{5}+1}{2} \simeq 1.618, \tag{2}$$

and satisfies the equation

$$\phi^2 = \phi + 1, \tag{3}$$

which implies that

$$\phi^{k+1} = \phi F_{k+1} + F_k, \tag{4}$$

where F_k are the Fibonacci numbers. [Zadar on the Dalmatian coast in Croatia is an ancient city with a rich history and a university whose origin dates back to 1396. One person who taught there was Luca Pacioli, whose famous work *Divina Proportione* (1509) was illustrated by Leonardo da Vinci.]

The character of each representation is its trace and must satisfy the following two orthogonality conditions

$$\sum_{C_i} n_i \chi_{ai} \chi^*_{bi} = n\delta_{ab}, \qquad \sum_{\chi_a} n_i \chi_{ai} \chi^*_{aj} = n\delta_{ij}, \tag{5}$$

where n is the total number of elements. The number of irreducible representations must be equal to the number of equivalence classes.

The two irreducible two-dimensional representations of D_5 may be chosen as follows. For **2**, let

$$\begin{aligned} &C_1 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_2 : \begin{pmatrix} 0 & \omega^k \\ \omega^{5-k} & 0 \end{pmatrix}, \ (k = 0,1,2,3,4); \\ &C_3 : \begin{pmatrix} \omega & 0 \\ 0 & \omega^4 \end{pmatrix}, \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega \end{pmatrix}, \quad C_4 : \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix}, \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^2 \end{pmatrix}, \end{aligned} \tag{6}$$

where $\omega = \exp(2\pi i/5)$, then $\mathbf{2}'$ is simply obtained by interchanging C_3 and C_4. Note that

$$2\cos(2\pi/5) = \phi - 1, \quad 2\cos(4\pi/5) = -\phi, \tag{7}$$

as expected.

For D_n with n prime, there are $2n$ elements divided into $(n+3)/2$ equivalence classes: C_1 contains just the identity, C_2 has the n reflections, C_k from $k=3$ to $(n+3)/2$ has 2 elements each of order n. There are 2 one-dimensional representations and $(n-1)/2$ two-dimensional ones. For $D_3 = S_3$, the above reduces to the "complex" representation with $\omega = \exp(2\pi i/3)$ discussed in a recent review [1].

The group multiplication rules of D_5 are

$$\mathbf{1}' \times \mathbf{1}' = \mathbf{1}, \quad \mathbf{1}' \times \mathbf{2} = \mathbf{2}, \quad \mathbf{1}' \times \mathbf{2}' = \mathbf{2}', \tag{8}$$

$$\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{2}', \quad \mathbf{2}' \times \mathbf{2}' = \mathbf{1} + \mathbf{1}' + \mathbf{2}, \quad \mathbf{2} \times \mathbf{2}' = \mathbf{2} + \mathbf{2}'. \tag{9}$$

In particular, let $(a_1, a_2), (b_1, b_2) \sim \mathbf{2}$, then

$$a_1 b_2 + a_2 b_1 \sim \mathbf{1}, \quad a_1 b_2 - a_2 b_1 \sim \mathbf{1}', \quad (a_1 b_1, a_2 b_2) \sim \mathbf{2}'. \tag{10}$$

Similarly, in the decomposition of $\mathbf{2}' \times \mathbf{2}'$, $(a'_2 b'_2, a'_1 b'_1) \sim \mathbf{2}$, and in the decomposition of $\mathbf{2} \times \mathbf{2}'$, $(a_2 a'_1, a_1 a'_2) \sim \mathbf{2}$, and $(a_2 a'_2, a_1 a'_1) \sim \mathbf{2}'$.

The most natural assignment of the 3 lepton families under D_5 is

$$(\nu_i, l_i), \; l_i^c \sim \mathbf{1} + \mathbf{2}. \tag{11}$$

Assuming two Higgs doublets $\Phi_1 \sim \mathbf{1}$, $\Phi_2 \sim \mathbf{1}'$, the charged-lepton mass matrix is then of the form

$$\mathcal{M}_l = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b-c \\ 0 & b+c & 0 \end{pmatrix}, \tag{12}$$

where a, b come from $\langle \phi_1^0 \rangle$, and c from $\langle \phi_1^0 \rangle$. Redefining $l_{2,3}^c$ as $l_{3,2}^c$, $\mathcal{M}_l$ becomes diagonal with $m_e = |a|$, $m_\mu = |b-c|$, $m_\tau = |b+c|$.

Assuming that neutrino masses are Majorana and that they come from the naturally small vacuum expectation values [2] of heavy Higgs triplets $\xi_1 \sim \mathbf{1}$, $\xi_{2,3} \sim \mathbf{2}$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix} \tag{13}$$

as advertised, where a, b come from $\langle \xi_1^0 \rangle$, and $c = f\langle \xi_3^0 \rangle$, $d = f\langle \xi_2^0 \rangle$. The two texture zeros are the result of the absence of a Higgs triplet transforming as $\mathbf{2}'$. In the case of $D_3 = S_3$, there is only one two-dimensional representation, hence these zeros cannot be maintained without also making $c = d = 0$.

The decomposition $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{2}'$ holds not only in D_5, but also in D_n with n prime and $n > 5$. For example in D_7, there are 3 two-dimensional irreducible representations, corresponding to the 3 cyclic permutations of

$$C_3 \quad : \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^6 \end{pmatrix}, \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^6 \end{pmatrix}, \tag{14}$$

$$C_4 \quad : \quad \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^5 \end{pmatrix}, \quad \begin{pmatrix} \omega^5 & 0 \\ 0 & \omega^2 \end{pmatrix}, \tag{15}$$

$$C_5 \quad : \quad \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^4 \end{pmatrix}, \quad \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^3 \end{pmatrix}, \tag{16}$$

where $\omega = \exp(2\pi i/7)$. It is clear that

$$\mathbf{2}_1 \times \mathbf{2}_1 = \mathbf{1} + \mathbf{1}' + \mathbf{2}_2, \quad \mathbf{2}_2 \times \mathbf{2}_2 = \mathbf{1} + \mathbf{1}' + \mathbf{2}_3, \tag{17}$$

etc. Hence Eq. (13) is valid in all these symmetries.

Phenomenologically, Eq. (13) has been studied [3] as an example of the class of neutrino mass matrices with two texture zeros. It was first derived from a symmetry (Q_8 or D_4) only recently [4]. Whereas Q_8 or D_4 allows other forms, D_n with n prime and $n \geq 5$ allows only Eq. (13). Models based on $D_4 \times Z_2$ have also been proposed [5]. The 4 parameters of Eq. (13) imply that $m_{1,2,3}$ are related to the mixing angles. Given the present global experimental constraints [6]

$$\Delta m^2_{atm} = (1.5 - 3.4) \times 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{atm} > 0.92, \tag{18}$$

$$\Delta m^2_{sol} = (7.7 - 8.8) \times 10^{-5} \text{ eV}^2, \quad \tan^2 \theta_{sol} = 0.33 - 0.49, \tag{19}$$

and $|\sin\theta_{13}| < 0.2$, the allowed region in the $m_3 - m_2$ plane has been obtained in Ref. [5]. That figure is reproduced here for the convenience of the reader. It shows that there are lower bounds on m_2 and m_3 and that $m_3 < m_2$ up to about 0.1 eV. The parameter a in Eq. (13) measures neutrinoless double beta decay and has a lower bound of about 0.02 eV in this case.

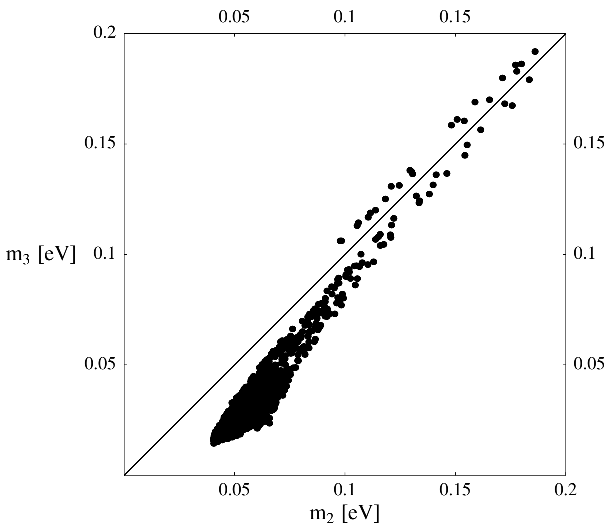

Fig. 1. Allowed region in $m_2 - m_3$ plane for Eq. (13).

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POLIGONSKI IZVOD MATRICE NEUTRINSKIH MASA

Predstavljanja grupe simetrija D_n pravilnog $n$tero-stranog poligona sadrže tvorbena pravila množenja ako je n primbroj. Primjenom D_n sa $n = 5$ ili većim, izvodi se dobro poznata matrica masa Majoraninih neutrina.