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# ON A NOVEL REGULARIZATION SCHEME

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We develop the mode-matching regularization scheme particularly suitable for problems in a uniform magnetic field. We apply this regularization scheme to bare QED vacuum in $(2 + 1)$ dimensions. This regularization scheme gives the exact renormalization of the bare QED vacuum and establishes its diamagnetic nature.

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## 1. Introduction

In any field theory, because of its infinite number of degrees of freedom, the zero-point energy diverges. In a conventional field theory, the infinite zero point energy is always discarded, since it can be reabsorbed in a suitable redefinition of the zero-point energy. This is justified in the sense that the infinite zero-point energy is unobservable. However, the change in zero-point energy caused by the external constraints is finite and observable. So, according to Casimir's idea [1], the physical vacuum energy can be defined as the difference between the zero-point energy corresponding to the vacuum configurations with constraints and the one corresponding to the free vacuum configurations. This definition must be supplemented in general with a regularization prescription in order to obtain a finite convergent expression. Various regularization schemes [2,3] have been developed in the field theory literature to obtain a finite cut-off-independent result. Here, we apply a simple yet interesting regularization scheme to bare QED case in two spatial dimensions. This regularization scheme [4] was previously used in the case of a charged scalar field in an external magnetic field in two spatial dimensions. The purpose of this paper is twofold. One is to establish this mode-matching regularization scheme as one of the important schemes suitable for an external magnetic field, and secondly, to establish the nature of the $(2+1)$ bare QED vacuum in an external magnetic field. In literature, the renormalization of QED vacuum in $(2+1)$ dimensions has been discussed in a background magnetic field in the effective action formalism [5]. This example apart from its simplicity illustrates an important fact that the naive $N$-particle nature does *not* always translate to a corresponding field theory.

This paper is organised as follows. In the next section, we develop the mode-matching regularization and apply it to the non-relativistic case. In Sect. 3, we apply it to the relativistic case, i. e. the bare QED vacuum, and show that the response of the vacuum is diamagnetic in nature. Finally, in Sect. 4, we give our conclusions.

## 2. *Regularization in non-relativistic case*

The regularization scheme we want to develop basically conserves the number of modes. Without the magnetic field, the phase-space coordinates of a particle are $(x, p_x, y, p_y)$. The spectrum is continuous. But in the presence of a magnetic field, the phase space is governed by $(x, y, \Pi_x, \Pi_y)$. The spectrum now becomes discrete and all energy levels are highly degenerate. This degeneracy of the system is related to the non-commutavity of $\Pi_x$ and $\Pi_y$. Since the commutation relation between $\Pi_x$ and $\Pi_y$ is gauge invariant, hence the mode matching is gauge invariant. Again, the phase-space density is invariant in any Lorentz frame. Therefore, this procedure can be justified on physical grounds.

Counting the modes up to the $L$-th Landau level, we find

$$\frac{eBA}{hc}\sum_{l=0}^{L} 1 = \frac{eBA}{hc}(L+1)\,. \tag{1}$$

Here, $A$ is the area of the system. Similarly, for the momentum cut-off up to $\Lambda$, we get the modes without the magnetic field as

$$\frac{2\pi A}{h^2}\int\limits_0^{\Lambda} p\,\mathrm{d}p = \frac{\pi A \Lambda^2}{h^2}\,. \tag{2}$$

According to this mode-matching principle, we get

$$\Lambda^2 = \frac{eBh}{\pi c}(L+1)\,. \tag{3}$$

This relation may look unphysical in the sense that $\Lambda$ is depending on the magnetic field. Instead, one should consider the mode-matching relation to find out $L$. Given a value of $\Lambda$, one can fix the $L$ value on the magnetic field. In this way one can compare the two free energies. For non-integer values of $L$, the Landau level sum is not defined. One can also consider this as an intermediate step rather than the final one. Of course, the physical quantity is here the difference between the two free energies (when the cutoff is taken to infinity), not the individual ones.

We have used a sharp cut-off to both the integral and the sum involved. Instead, one can also use the regularising cut-off function [6] like

$$\exp(-(p^2 + m^2)/(2\Lambda^2))$$

with a large $\Lambda$. With this regularization function, we can keep the limits of integration unaltered. This cut-off function is a smooth function of $p$ with different weights to different momenta $p$. Contributions from the high values of the momentum $p$ are not significant. Again, using the mode-matching principle, we can get a relation among $L$ and $\Lambda$. Formally, one can use this scheme to extract the universal component of the vacuum contribution; however, its nontrivial non-linear structure makes the calculation a little bit difficult for an exact analysis.

The non-relativistic expression of energy in the presence of a uniform magnetic field is given by

$$E(B) = \frac{eBA}{hc} \sum_{l=0}^{\infty} \left( l + \frac{1}{2} \right) \hbar\omega_c \,, \tag{4}$$

and the energy without the magnetic field is

$$E(0) = \frac{A}{h^2} \int_{-\infty}^{\infty} \mathrm{d}p_x \ \mathrm{d}p_y \ \ \frac{1}{2m}(p_x^2 + p_y^2)\,. \tag{5}$$

It is quite obvious that these two energies are divergent, so we use now the mode-matching regularization scheme to obtain a finite difference between them. Now, the energy in the presence of a magnetic field up to the $L$-th Landau level is given by

$$E(B, L) = \frac{e^2 B^2 A}{4\pi m c^2}(L + 1)^2 \,. \tag{6}$$

The energy without the magnetic field with a momentum cutoff $\Lambda$ is given by

$$E(0, \Lambda) = \frac{\pi A \Lambda^4}{4mh^2} \,. \tag{7}$$

Now, due to the mode matching, we notice that

$$E(B, L) = E(0, \Lambda)\,. \tag{8}$$

Equation (8) shows the exact cancellation of the two infinities in the case of the non-relativistic limit. This also restores the property of a true vacuum which has zero energy. One can also use the same mode-matching relation to $(3+1)$ dimensions to obtain the zero vacuum energy.

Up till now, we have not taken into account the spin of the electron. In the presence of the spin, the non-relativistic energy levels in a uniform magnetic field are given by

$$E_{l,\sigma} = \left(l + \frac{1}{2}\hbar\omega_c\right) - \sigma\hbar\omega_c \,, \tag{9}$$

where $\sigma$ is the spin index which can take the values $\pm\frac{1}{2}$, and $\omega_c = eB/(mc)$ is the cyclotron frequency. Because of the degeneracy between the spin-up and spin-down electrons, Eq. (3) becomes

$$\Lambda^2 = \frac{eB\hbar}{c}(2L+1)\,. \tag{10}$$

Now, the energy in presence of the magnetic field is given by

$$E_0(B,L) = \frac{eBA}{hc}\left[\sum_{l=0}^{L} l\hbar\omega_c + \sum_{l=0}^{L-1}(l+1)\hbar\omega_c\right]. \tag{11}$$

After simplification, we obtain

$$E_0(B,L) = \frac{e^2B^2A}{2\pi mc^2}\left[(L+\frac{1}{2})^2 - \frac{1}{4}\right]. \tag{12}$$

Similarly, the energy without the magnetic field in this case becomes

$$E_0(0,\Lambda) = \frac{\pi A\Lambda^4}{2mh^2}\,. \tag{13}$$

Substituting the value of $\Lambda$, we get

$$E_0(0,L) = \frac{e^2B^2A}{2\pi mc^2}\left(L+\frac{1}{2}\right)^2. \tag{14}$$

Now, comparing the two energies, it is evident that

$$E_0(B,L) - E_0(0,L) = -\frac{e^2B^2A}{8\pi mc^2}\,. \tag{15}$$

The difference is seen to be finite and independent of the cutoff. Also, the difference decreases with the magnetic field. This in turn suggests that the ground state of electrons having spin $1/2$ degrees of freedom is paramagnetic in nature.

## *3. Regularization of QED vacuum*

In this section, we follow the natural units $\hbar = 1$ and $c = 1$. In the case of the bare QED vacuum in an external homogeneous magnetic field, the single-particle energy levels for spin-up and spin down states are given by [7]

$$\omega_{l,\sigma} = \sqrt{m^2 + 2l\ eB},\ \omega_{l,-\sigma} = \sqrt{m^2 + (2l+2)\ eB}\,. \tag{16}$$

The total energy of the system is given by

$$E_0(B) = -\frac{eBA}{2\pi} \sum_{l=0}^{\infty} (\omega_{l,\sigma} + \omega_{l,-\sigma})\ . \tag{17}$$

(As noted above, $A$ is the area of the system.) Note that there is a difference in sign relative to the vacuum energy in the spin-zero case. This negative sign occurs because spin 1/2 particles satisfy the anticommutation relation rather than the commutation one [8,6,9]. The energy of spin-1/2 particle without the magnetic field is given by

$$E_0(0) = -2 \times \frac{A}{2\pi} \int_0^{\infty} p\,\mathrm{d}p\ \sqrt{p^2 + m^2}\,. \tag{18}$$

The factor 2 comes from the spin degeneracy. Again, we note that the energy diverges both in the presence of an external magnetic field and without it. Below we show that their difference is finite and positive in the asymptotic limit. We regularize the energy in the presence of the magnetic field with a finite Landau level cut-off $L$, and the energy then becomes

$$E_0(B, L) = -\frac{eB\,A}{2\pi} \left[ \sum_{l=0}^{L}\ \omega_{l,\sigma} + \sum_{l=0}^{L-1} \omega_{l,-\sigma} \right]\ . \tag{19}$$

The energy without the magnetic field with a finite momentum cut-off $\Lambda$ is given by

$$E_0(0, \Lambda) = -\ \frac{2A}{6\pi}\ \left[ (\Lambda^2 + m^2)^{\frac{3}{2}} - m^3\ \right]\ . \tag{20}$$

However, to compare these two energies (19) and (20), we need another relation among $L$ and $\Lambda$. This is provided by the mode-matching regularization scheme. The number of modes in the presence of an external homogeneous magnetic field with a cut-off $L$ is given by

$$\frac{eBA}{2\pi} \left[ \sum_{l=0}^{L}\ 1 + \sum_{l=0}^{L-1} 1 \right] = \frac{eBA}{2\pi} (2L+1)\,. \tag{21}$$

Similarly, the number of modes without the magnetic field with a finite momentum cut-off $\Lambda$ is given by

$$\frac{2A}{2\pi}\int_0^\Lambda p\,\mathrm{d}p = \frac{A\Lambda^2}{2\pi}\,. \tag{22}$$

Equating these, we get a relation between $L$ and $\Lambda$ as

$$\Lambda^2 \;=\; (2L+1)eB\,. \tag{23}$$

Therefore, Eq. (20) becomes

$$E_0(0,\Lambda) = -\frac{2A}{6\pi}\left[((2L+1)eB+m^2)^{3/2}-m^3\right]\,. \tag{24}$$

It can be easily shown by numerically plotting and comparing the two energies that

$$E_0(B,L) \geq E_0(0,L)\,. \tag{25}$$

Before we go on to prove the (cut-off independent) renormalization and positiveness of the free-energy difference, we present a simple table obtained numerically which illustrates effectively the renormalization. The parameters used in following table are $m = 0.1$ and $B = 2$. Here, $\Delta E = (2\pi/A)\,[E_0(B,L) - E_0(0,\Lambda)]$.

*TABLE 1. Free-energy difference, $\Delta E$, for different values of cutoffs ($L$).*

| Different values of cutoffs ($L$) | $\Delta E$ |
|---|---|
| $L = 10$ | 1.71448 |
| $L = 100$ | 1.68021 |
| $L = 1000$ | 1.68623 |
| $L = 9000$ | 1.68895 |

The above table illustrates two points, first, the difference is *positive* and hence the response is diamagnetic. And secondly, the difference is going to saturate with the cut-off and becomes independent of the cut-off in the asymptotic limit. For the sake of completeness, we provide an analytical proof below. It is easy to notice that the difference between the two energies can be written as

$$\Delta E_0(B,m) = \\ 2eBA\sum_{l=0}^{\infty}\left[\int_0^1 \mathrm{d}\alpha\sqrt{m^2+2(l+\alpha)eB}-\sqrt{m^2+2leB}-\sqrt{m^2+(2l+2)eB}\right]. \tag{26}$$

Now, introducing the dimensionless variable $\rho = eB/m^2$, we can write the above difference in the dimensionless form as

$$g(\rho) = \frac{\Delta E_0(B,m)}{2\sqrt{2}\rho^{3/2}m^3A} = \sum_{l=0}^{\infty} d_l(B,m)\,, \tag{27}$$

where $d_l(B,m)$ is given by

$$d_l(B,m) = 2\int_0^1 \mathrm{d}\alpha\ \sqrt{z_l+\alpha} - (\sqrt{z_l}\ + \sqrt{z_l+1})\,, \tag{28}$$

where $z_l = (1+2l\rho)/2\rho$. Now, note that the function $f(\alpha) = \sqrt{z_l+\alpha}$ is convex (see Fig. 1).

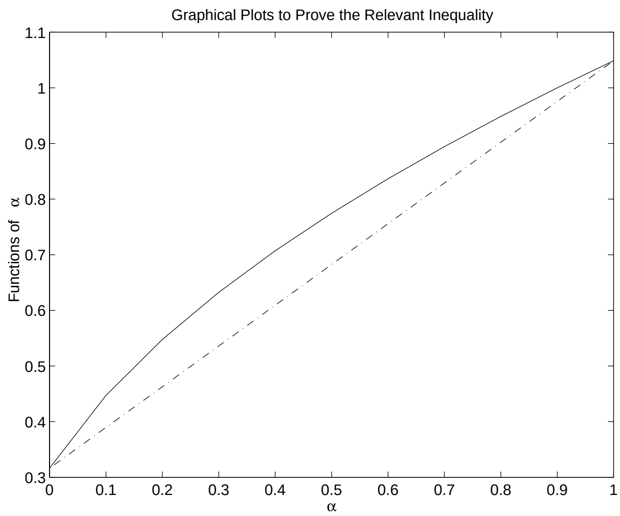


*Fig. 1. The curve is drawn for the function $f(\alpha) = (0.1+\alpha)^{1/2}$. The dash-dot line is the chord joining the two end points of the curve. The area under the curve is more than that under the chord. Hence, the positivity of $d_l(B,m)$ is proved.*

Hence, it follows that

$$\int_0^1 \mathrm{d}\alpha\ \sqrt{z_l+\alpha} - \frac{\sqrt{z_l}\ + \sqrt{z_l+1}}{2} \geq 0\,. \tag{29}$$

The convergence of this function can be proved easily. We note that

$$\int_0^1 \mathrm{d}\alpha\ f(\alpha)\ \leq \left[f(1/2)\ -\ \frac{f(0)+f(1)}{2}\right]\,. \tag{30}$$

Now, applying mean value theorem twice, it is easy to show that

$$\int_0^1 \mathrm{d}\alpha\ f(\alpha) \leq \frac{1}{16(z_l+\alpha)}3/2\,. \tag{31}$$

Notice that the difference between the free energies varies with the cutoff as $1/\sqrt{L}$. This shows that in the relativistic case, the response of QED vacuum is diamagnetic in nature.

Instead of using the previous mode matching (23), one can use the energy of the highest state to obtain relations like $\Lambda^2 = 2LeB$ or $\Lambda^2 = (2L+2)eB$. It turns out that the energy difference becomes *cut-off* dependent. Therefore, this regularization scheme based on the matching of energy can be discarded.

## 4. Conclusion and discussion

The diamagnetism of QED vacuum in (3+1) dimension have also been discussed in Ref. [9]. Here, we also find the same response as QED in (2+1) dimension. However, the result obtained in Ref. [9] depends on the cutoff used in the theory. Our result uses a proper regularization scheme to deduce the diamagnetic nature of the renormalised vacuum. If one thinks in terms of virtual pairs of electrons and positrons which have spin, then one would naively believe that the vacuum will be paramagnetic in nature. However, it turns out that it is diamagnetic. This behaviour is well known in literature [9,10] and has been explained as a consequence of the Pauli exclusion principle. The nature of the QED vacuum in an *inhomogeneous* magnetic field as well as with a finite chemical potential and finite fermion density has been discussed recently in the literature [11,12]. To summarize, we have shown the renormalization of bare QED vacuum in two spatial dimensions in an external homogeneous magnetic field and established that the energy in presence of an external magnetic field is higher than without the magnetic field. This inequality has been proved *exactly* through a novel regularization scheme.

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## NOVA REGULACIJSKA SHEMA

Razvili smo regulacijsku shemu s usklađivanjem modova koja je posebno pogodna za probleme u jednolikom magnetskom polju. Primijenili smo je u prostoru s $(2+1)$ dimenzija i čistom QED vakuumu. Ova regulacijska shema daje egzaktnu renormalizaciju čistog vakuuma i utvrđuje njegovu dijamagnetsku narav.