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INHERENTLY RELATIVISTIC QUANTUM THEORY
Part II. CLASSIFICATION OF SOLUTIONS

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Dedicated to Professor Kseno Ilakovac on the occasion of his 70$^{\text{th}}$ birthday

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The abstract quantal algebra developed in Part I of the present work describes the common structure of the two known mechanics, classical and quantum. By itself, however, it is not physics. It is a mathematical object, or, as some might say, it is only mathematics, a valid objection if quantal algebra were meant to be an end in itself, for physics is not in abstract theories, but in their concrete realizations. Hence, the immediate question is whether at least one new concrete realization of the quantal algebra exists, for it is among these that a physically valid generalization of quantum mechanics might be found. The search for all realizations of an abstract theory is known in mathematics as *structure theory*, or *the classification problem.* Usually difficult, it is relatively easy in our case because the foundations have already been laid in Cartan's classification of the semi-simple Lie algebras. Since the quantal algebra contains a Lie algebra, we only need to adapt the standard work to our case by imposing some additional conditions. The result is that the semi-simple quantal algebra has exactly two realizations. Expressed in terms of groups, one is the infinite family of unitary groups, $SU(n)$, (i.e., standard quantum mechanics), the other is an exceptional solution, the group $SO(2,4)$. Classical mechanics does not appear as a solution because the requirement of semi-simplicity eliminates the canonical group. Thus, if quantum mechanics can be generalized, the generalization is somehow related to the group $SO(2,4)$, and as this group contains the relativistic space-time structure, it appears that an inherently covariant generalization might be possible.

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1. Introduction

Trusting the constructive power of mathematical abstraction, which, when successful, leads to concrete results not envisaged initially, we developed in Part I the "quantal algebra" $\{\mathcal{O},\sigma,\alpha\}$ as an abstraction of a single non-degenerate prototype (quantum mechanics) and of its degenerate limit (classical mechanics). The restriction to semi-simple quantal algebras eliminates classical mechanics. In the present chapter we harvest the returns of this abstraction by deriving all concrete realizations of semi-simple quantal algebras by an exhaustive classification procedure. This step is essential, as physics does not reside in abstract structures, but in their realizations. It is the latter that ultimately allow numerical calculations. We shall prove that these concrete realizations are standard quantum mechanics and the exceptional solution $SO\,(2,4)$.

The historical precursor of the classification we develop in the present article is the Killing-Cartan classification of semi-simple Lie algebras — also referred to as their "structure theory". It begins with the theorem which states that the indecomposable building blocks of all Lie groups are the semi-simple ones, so that classification applies only to the Lie algebras of the latter. Extending Cartan's procedure, we shall similarly classify the realizations of semi-simple quantal algebras.

Since, by definition (Part I, Section 3), a semi-simple quantal algebra $\{\mathcal{O},\sigma,\alpha\}$ contains a semi-simple Lie subalgebra, $\{\mathcal{O},\alpha\}$, the approach to classification suggests itself. It consists in expanding Cartan's structure theory of semi-simple Lie algebras by including the product σ and its associated Leibnitz and association identities. To avoid repeatedly referencing equations in Part I, we list here the identities defining a quantal algebra. They are, respectively, the Jacobi, Leibnitz and association identities:

$$(f\alpha g)\,\alpha h + (g\alpha h)\,\alpha f + (h\alpha f)\,\alpha g = 0 \tag{1}$$

$$h\alpha\,(f\sigma g) = (h\alpha f)\,\sigma g + (h\alpha g)\,\sigma f \tag{2}$$

$$[f,g,h] = ag\alpha\,(h\alpha f) \tag{3}$$

where

$$[f,g,h] = (f\sigma g)\,\sigma h - f\sigma\,(g\sigma h)$$

2. Cartan's classification

In this section we review the terminology and theorems of Lie algebra which will be needed for the classification of quantal algebras, formulating them for immediate applicability in the abstract notation defined in Part I. They can all be found in standard references, for example [1–3]. Hence, the proofs need not be repeated, though a few short ones are, as an introduction to work in the notations of abstract quantal algebra.

Given a set $\{e_a\}$ of basis vectors in the underlying linear space $\mathcal{O}$ of a Lie algebra $\{\mathcal{O}, \alpha\}$, the product α is represented by a set of **structure constants**, C^c_{ab}, defined by the relations

$$e_a \alpha e_b = C^c_{ab} e_c\,, \tag{4}$$

with the usual summation convention. The antisymmetry and Jacobi identities imply

$$C^c_{ab} + C^c_{ba} = 0, \tag{5}$$

$$C^m_{ab} C^k_{cm} + C^m_{ca} C^k_{bm} + C^m_{bc} C^k_{am} = 0. \tag{6}$$

To every pair of elements $f, g \in \mathcal{O}$ is associated an invariant scalar. To construct it, one uses *the adjoint operator* $\hat{f}$ defined as

$$\hat{f} \stackrel{\text{def}}{=} f\alpha\,.$$

Given n basis vectors in the Lie algebra $\mathcal{O}$, the operator $\hat{f}$ is represented by an $n \times n$ real matrix. The product $\hat{f}\hat{g}$ of two adjoint operators is to be understood as the product of the matrices that represent them. The trace of all such products defines a scalar product in $\mathcal{O}$, referred to as the Cartan metric.

Definition 1 *The* ***Cartan metric*** *in* $\mathcal{O}$ *is defined as*

$$(f, g) \stackrel{\text{def}}{=} \mathrm{Tr}\left(\hat{f}\hat{g}\right)\,. \tag{7}$$

Clearly, $(f, g) = (g, f)$, due to the fact that the trace of a product of matrices is cyclically symmetric.

In terms of structure constants, the corresponding metric tensor reads

$$g_{ab} = C^m_{ak} C^k_{bm}. \tag{8}$$

One easily verifies that the scalar product (f, g) is invariant under all infinitesimal transformations, $f \to f + \varepsilon h \alpha f$, generated by an arbitrary element $h \in \mathcal{O}$:

$$(f + \varepsilon h\alpha f, g + \varepsilon h \alpha g) = (f, g)\,.$$

Definition 2 *A Lie algebra is said to be* ***semi-simple*** *if it contains no subalgebra* $\mathcal{A} \in \mathcal{O}$ *such that* $\mathcal{A}\alpha\mathcal{A} = \{0\}$ *and* $\mathcal{A}\alpha\mathcal{O} \subseteq \mathcal{A}$..

Theorem 1 *(Cartan) A Lie algebra is semi-simple if and only if its Cartan metric is non-singular, i.e., if*

$$\det\left(g_{AB}\right) \neq 0. \tag{9}$$

The concept that plays a key role in the classification of semi-simple Lie algebras is that of a maximal Abelian subalgebra, also called a **Cartan subalgebra**. It is unique in the following sense:

Theorem 2 *(Cartan) All maximal Abelian subalgebras of a semi-simple Lie algebra are mutually isomorphic.*

The practical implication of this theorem is that one can select an arbitrary Cartan subalgebra to develop the classification procedure, knowing that the conclusions, structurally speaking, are independent of the selection.

If one denotes a Cartan subalgebra by $\mathcal{O}_0$ and its complement by $\mathcal{O}_1$, the linear space of observables consists of three parts,

$$\mathcal{O} = \mathcal{O}_0 \oplus \mathcal{O}_1 \oplus e\mathbb{R}.$$

The dimension of the Cartan subalgebra, $l = \dim(\mathcal{O}_l)$, is referred to as the **rank** of the Lie algebra. The dimension of the Lie algebra (the space $\mathcal{O}$ without the unit e) is $r = \dim(\mathcal{O}_0 \oplus \mathcal{O}_1)$. Hence, $\dim(\mathcal{O}_1) = r - l$.

The proofs of theorems in Cartan's structure theory rely on the eigenvalues and eigenvectors of the adjoint operators $\hat{h}_i = h_i\alpha$ associated to the elements $h_i \in \mathcal{O}_0$. Clearly, for all i, j,

$$h_i \alpha h_j = 0. \tag{10}$$

The eigenvectors play an essential role due to the following theorem:

Theorem 3 *All elements of a Cartan subalgebra $\mathcal{O}_0$ share the same eigenrays.*

Proof. Let ρe_ρ be an eigenray of some arbitrary element $h \in \mathcal{O}_0$ (ρ being the eigenvalue and e_ρ the eigenvector),

$$h\alpha e_\rho = \rho\, e_\rho\,.$$

With another arbitrary element, $h' \in \mathcal{O}_0$, apply the operator $h'\alpha$ to both sides of this relation and use relation (10) to interchange h and h',

$$h'\alpha\,(h\alpha e_\rho) = h\alpha\,(h'\alpha e_\rho) = \rho\, h'\alpha e_\rho$$

which proves that $\rho h'\alpha e_\rho$ is also an eigenray of h. □

Once a Cartan subalgebra $\mathcal{O}_0$ has been selected, a linear basis of eigenvectors is uniquely defined in its $(r-l)$-dimensional complementary subspace $\mathcal{O}_1$. This basis decomposes the space $\mathcal{O}_1$ into a direct sum of invariant rays. The notation used for the eigenvectors is e_α, where Greek indices runs over a set of $r - l$ values.

Let $\{h_1, h_2, ..., h_l\}$ denote some linear basis in the Cartan subalgebra $\mathcal{O}_0$. To every eigenvector e_β is then associated an ordered set of eigenvalues, referred to as a **root vector**, or, simply, a **root**:

$$\beta \overset{\text{def}}{=} \{\beta_1, \beta_2, \cdots, \beta_l\},$$

whose components are defined by the l characteristic equations

$$h_j \alpha e_\beta = \beta_j e_\beta. \tag{11}$$

Clearly, there are $r - l$ root vectors.

Theorem 4 *There are no degenerate root vectors, i.e., $e_\alpha \neq e_\beta$ implies $\alpha \neq \beta$, meaning that the sets of root vector components $\{\alpha_1, \alpha_2, \cdots, \alpha_l,\}$, $\{\beta_1, \beta_2, \cdots, \beta_l\}$, differ in at least one component.*

Hence, the Greek indices which label the eigenvectors, e.g., α in e_α, represent root vectors.

Definition 3 *The **root space**, V^l, is the l-dimensional linear space spanned by all root vectors.*

The root space inherits the Cartan metric,

$$(\alpha, \beta) \overset{\text{def}}{=} (e_\alpha, e_\beta), \tag{12}$$

which, by Cartan's theorem 1, is non-singular. In the basis $\{h_1, h_2, \cdots, h_l\}$, the corresponding metric tensor is

$$g_{ij} = C^l_{ik} C^k_{jl}. \tag{13}$$

Theorem 5 *There are no isotropic root vectors.*

This means that for every root vector $\alpha \in V^l$, $(\alpha, \alpha) = 0$ implies $\alpha = 0$, i.e., $\alpha^i = 0$ for all $i = 1 \cdots l$.

To classify the structure constants C^a_{bc}, Cartan's approach eliminates the arbitrariness they inherit from the arbitrariness of the coordinate system by working in a basis of eigenvectors, the latter being uniquely defined by the Cartan subalgebra $\mathcal{O}_0$.

Theorem 6 *The eigenvectors are paired by a duality relation — dual pairs having root vectors whose sum vanishes.*

This theorem implies $r - l = even$, i.e., the space $\mathcal{O}_1$ is even-dimensional. The notation used for the pairs of dual eigenvectors is $\{e_\alpha, e_{-\alpha}\}$.

The Lie products we shall need are

$$h_i \alpha \, e_\alpha = \alpha_i e_\alpha, \tag{14}$$

$$h_i \alpha \, e_{-\alpha} = -\alpha_i e_{-\alpha}, \tag{15}$$

$$h_\alpha \stackrel{\text{def}}{=} e_\alpha \alpha \, e_{-\alpha} = \alpha^k h_k. \tag{16}$$

As the products $e_\alpha \alpha \, e_\beta$ for $\alpha + \beta \neq 0$ will not be needed to complete the classification of quantal algebras, we do not review their properties.

We see from relations (14), (15) and (16), that the root vectors play the role of structure constants. We also note that for every root α the triplet $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ is a simple Lie algebra, i.e., that it has no invariant subalgebra. This can be seen from the multiplication table

α	h_ρ	e_ρ	$e_{-\rho}$
h_ρ	0	$(\rho^k \rho_k)\, e_\rho$	$-(\rho^k \rho_k)\, e_{-\rho}$
e_ρ	$-(\rho^k \rho_k)\, e_\rho$	0	$\rho^k h_k.$
$e_{-\rho}$	$(\rho^k \rho_k)\, e_{-\rho}$	$-\rho^k h_k$	0

The final concept required to complete the classification is that of a **string of roots:**

Definition 4 *If $\alpha, \beta \in V^l$ are any two root vectors, the set of root vectors of the form $\beta + k\alpha \in V^l$, where k is an integer, is called an α-**string containing** β.*

If the dimension r of the Lie algebra is finite, any string of roots is obviously finite, but what is remarkable — and essential to Cartan's procedure — is that *the maximal string length is fixed*, in other words, it does not depend on the dimension of the Lie algebra. Specifically,

Theorem 7 *A string of root vectors can have at most four elements, and they are all adjacent.*

Cartan's classification of solutions stems from the conditions this limitation imposes on the geometric relationships among root vectors.

3. Classification of Quantal Algebras

While the classification of semi-simple Lie algebras stems from the Jacobi identity alone, quantal algebras are also constrained by the Leibnitz and association identities. These additional conditions are satisfied by only some of the Cartan classes of Lie algebras. We identify them in the present section by strengthening Theorem 7.

All proofs follow essentially the same pattern. We start with triplets of eigenvectors assumed to satisfy the conditions (10) and (14) to (16) — which conditions already account for the Jacobi identity (1). We then introduce the commutative product σ and impose the Leibnitz identity (2) and the association identity (3). Expansion of both sides of each new relation using the appropriate characteristic equations leads to additional relations among the root vectors. Only those semi-simple Lie algebras which satisfy these relations are candidate semi-simple quantal algebras. That they actually are quantal algebras can be verified in two ways. One approach (rather laborious) would consist in completing Cartan's structure theory in the general formalism of the classification by verifying the Leibnitz and association identities not only for the products (14) to (16), but also for the products $e_\alpha \alpha e_\beta$ when $\alpha + \beta \neq 0$. A much simpler approach, taken in Part III of the present work, consists in verifying the relevant identities in a formalism best adapted to the relevant realizations.

The association identity, (3), contains the composition class parameter a, which, in principle, may be ± 1 or 0. We shall consider all three possibilities.

Proceeding by cases, we classify all relevant triplets into two types: those from the Cartan subalgebra $\mathcal{O}_0$, i.e. $\{h_i, h_j, h_k\}$, and those that also include elements from the space $\mathcal{O}_1$, but not to the point of maximal generality, $\{e_\alpha, e_\beta, e_\gamma\}$, which happens to be unnecessary for our purposes.

Triplets in the Cartan subalgebra

Taking the Cartan algebra triplet $\{h_i, h_j, h_k\}$, we expand the corresponding Leibnitz and association identities. The Leibnitz identity

$$h_i \alpha \left(h_j \sigma h_k\right) = \left(h_i \alpha h_j\right) \sigma h_k + h_i \sigma \left(h_j \alpha h_k\right)$$

implies $h_i \alpha \left(h_j \sigma h_k\right) = 0$, since each of the two summands on the right hand side vanishes. Hence, $h_j \sigma h_k$ is a constant or belongs to $\mathcal{O}_0$. In general, $h_j \sigma h_k \in e\mathbb{R} \oplus \mathcal{O}_0$, so that there exist two sets of real invariant coefficients, A_{ij}, S_{ij}^k, such that $h_i \sigma h_j = A_{ij} e +\ S_{ij}^k h_k$. Since the only invariant 2-component tensor is the Cartan metric restricted to the subalgebra $\mathcal{O}_0$, we have $A_{ij} = A g_{ij}$ for some $A \in \mathbb{R}$. Hence,

$$h_i \sigma h_j = A g_{ij} e + S_{ij}^k h_k, \tag{17}$$

where g_{ij} is defined by relation (13).

The coefficients S_{ij}^k are symmetric, $S_{ij}^k = S_{ji}^k$, due to the symmetry of the product σ. We next derive the conditions which are to be satisfied by A and S_{ij}^k.

Substitution of the expression (17) for the products sigma in the association identity

$$[h_i, h_j, h_k] = \left(h_i \sigma h_j\right) \sigma h_k - h_i \sigma \left(h_j \sigma h_k\right) = a h_j \alpha \left(h_k \alpha h_i\right) = 0,$$

yields two relations.

The first relation,

$$A\left(S_{ijk}-S_{kij}\right)=0, \tag{18}$$

implies either $A=0$, or, together with the symmetry in the first two indices, the complete symmetry of the tensor S_{ijk},

$$S_{ijk}=S_{kij}=S_{jki}=S_{jik}=S_{kji}=S_{ikj}. \tag{19}$$

We first consider the general case, $A\neq 0$, $S_{ijk}\neq 0$.

It will prove convenient to define the vector S_i,

$$S_i=S_{ik}^{k}, \tag{20}$$

as the trace of the tensor S_{ij}^{k}.

The second relation, after taking the total symmetry (19) into account, reads

$$S_{ij}^{r}S_{kmr}-S_{kj}^{r}S_{imr}=A\left(g_{kj}g_{im}-g_{ij}g_{km}\right). \tag{21}$$

Simpler equations follow by taking traces (a reminder: $g^{ij}g_{ij}=l$, the dimension of the Cartan subalgebra):

$$S_{i}^{mr}S_{jmr}-S_{ij}^{r}S_{r} = A\left(l-1\right)g_{ij}, \tag{22}$$

$$S^{ijk}S_{ijk}-(S,S) = Al\left(l-1\right). \tag{23}$$

Equations (19) and (22) exhaust what can be learned from triplets in the Cartan subalgebra.

Mixed triplets

The triplets $T_1\overset{\text{def}}{=}\{h_i,h_j,e_\alpha\}$, $T_2\overset{\text{def}}{=}\{h_i,e_\alpha,e_{-\alpha}\}$ and $T_3\overset{\text{def}}{=}\{e_\alpha,e_\alpha,e_{-\alpha}\}$ are the only ones we need to consider, as they alone yield sufficiently strong necessary conditions to classify the semi-simple quantal algebras. That these conditions are also sufficient will be shown in Part III by direct construction.

The Leibnitz identity can be applied within the triplet T_1 in two ways. The first ordering yields

$$h_i\alpha\left(h_j\sigma e_\alpha\right)=h_j\sigma\left(h_i\alpha e_\alpha\right),$$

which, with relation (14), leads to

$$h_i\alpha\left(h_j\sigma e_\alpha\right)=\alpha_i\left(h_j\sigma e_\alpha\right).$$

Thus, $h_j\sigma e_\alpha$ is an eigenvector of the operator $h_i\alpha$ for arbitrary i, and the corresponding eigenvalue is α_i. By relation (14), this implies that $h_j\sigma e_\alpha$ is proportional to e_α, i.e., there exists a vector $\tilde{\alpha}_j\in V^l$ such that

$$h_j\sigma e_\alpha=\tilde{\alpha}_j e_\alpha\,. \tag{24}$$

It follows that the eigenvectors of the l operators $h_i\alpha$ are also the eigenvectors of the operators $h_i\sigma$, but with different eigenvalues, i.e., different "root vectors". We use the same notation for the new root vectors, but with a tilde. We refer to them as the **reciprocal root vectors**. Relation (24) is the sigma counterpart of relation (14).

By applying the association identity to the triplet T_2, one obtains

$$(e_{-\alpha}\sigma h_j)\,\sigma e_\alpha - e_{-\alpha}\sigma\,(h_j\sigma e_\alpha) = a h_j\alpha\,(e_{-\alpha}\alpha e_\alpha)\,.$$

By relations (16) and (10), the right hand side vanishes. With relation (24), this further implies

$$(h_j\sigma e_{-\alpha})\,\sigma e_\alpha = \tilde{\alpha}_j e_{-\alpha}\sigma e_\alpha.$$

Since, by relation (24),

$$h_j\sigma e_{-\alpha} = (-\alpha_j)^{\tilde{}}\; e_{-\alpha},$$

it follows that the reciprocal root vectors for the dual eigenvectors e_α and $e_{-\alpha}$ are equal, i.e., $(-\alpha_j)^{\tilde{}} = \tilde{\alpha}_j$. Hence, the relation corresponding to (15) is

$$h_j\sigma e_{-\alpha} = \tilde{\alpha}_j e_{-\alpha}. \tag{25}$$

Unlike the set of l operators $h_i\alpha$, which, by Theorem 4, has no degenerate eigenvalues (root vectors), the eigenvalues of the set of operators $h_j\sigma$ (the reciprocal root vectors) are at least doubly degenerate, since the eigenvectors e_α and $e_{-\alpha}$ belong to the same reciprocal root $\tilde{\alpha}_j$ (we say "at least" for lack of proof so far that all vectors $\tilde{\alpha}_j$ are different — but we shall soon see that they are).

A relationship between the root vectors and the structure constants S_{ijk} follows from the Leibnitz identity applied within the triplet T_1 in the second ordering of variables,

$$e_\alpha\alpha\,(h_i\sigma h_j) = (e_\alpha\alpha h_i)\,\sigma h_j + h_i\sigma\,(e_\alpha\alpha h_j)\,.$$

Substitution into this equation of relations (17) and (14) yields

$$e_\alpha\alpha\left(Ag_{ij}e + S_{ij}^{k}h_k\right) = (-\alpha_i e_\alpha)\,\sigma h_j + h_i\sigma\,(-\alpha_j e_\alpha)\,,$$

which simplifies to

$$\alpha^k S_{ijk} = \alpha_i\tilde{\alpha}_j + \tilde{\alpha}_i\alpha_j. \tag{26}$$

Contraction of both sides of relation (26) by α^j yields

$$\alpha^j\alpha^k S_{ijk} = (\alpha,\tilde{\alpha})\,\alpha_i + (\alpha,\alpha)\,\tilde{\alpha}_i, \tag{27}$$

while the trace (i.e., contraction by the metric tensor g^{ij}) is

$$(\alpha,S) = 2\,(\alpha,\tilde{\alpha})\,. \tag{28}$$

A relation analogous to (16) for the reciprocal vectors follows from the Leibnitz identity applied to the triplet T_2:

$$\begin{aligned}(h_i\sigma e_\alpha)\,\alpha e_{-\alpha} &= h_i\sigma\,(e_\alpha\alpha e_{-\alpha}) + (h_i\alpha e_{-\alpha})\,\sigma e_\alpha, \\ \tilde\alpha_i e_\alpha\alpha e_{-\alpha} &= h_i\sigma\left(\alpha^j h_j\right) + (-\alpha_i e_{-\alpha})\,\sigma e_\alpha, \\ \tilde\alpha_i\alpha^k h_k &= \alpha^j\left(A g_{ij} e + S^k_{ij} h_k\right) - \alpha_i\,(e_{-\alpha}\sigma e_\alpha)\,.\end{aligned}$$

Contracting both sides of the last relation by α^i, using relation (27), and dividing by (α,α) (which is allowed by theorem 2), yields

$$e_\alpha\sigma e_{-\alpha} = \tilde\alpha^k h_k + Ae. \tag{29}$$

This relation is analogous to (16), except that it has an additional constant term, Ae.

We next apply the Leibnitz identity to the triplet T_3,

$$e_\alpha\alpha\,(e_\alpha\sigma e_{-\alpha}) = e_\alpha\sigma\,(e_\alpha\alpha e_{-\alpha})\,,$$

which, with relations (29) and (16), yields

$$\begin{aligned}e_\alpha\alpha\left(\tilde\alpha^k h_k\right) &= e_\alpha\sigma\left(\alpha^k h_k\right), \\ -\,(\tilde\alpha,\alpha) &= (\tilde\alpha,\alpha)\,.\end{aligned}$$

The last equation implies

$$(\tilde\alpha,\alpha) = 0. \tag{30}$$

Hence, the Cartan root vectors and the reciprocal root vectors are mutually orthogonal. It then follows from relation (27) that the reciprocal root vectors are defined by the Cartan root vectors and the tensor S_{ijk} by the formula

$$\tilde\alpha_i = \frac{1}{(\alpha,\alpha)}\alpha^j\alpha^k S_{ijk}. \tag{31}$$

From relation (28) follows $(S,\alpha) = 0$. Since this equation holds for all root vectors, and since these vectors span the space V^l, the vector $S^i \in V^l$ vanishes,

$$S^i = 0. \tag{32}$$

This result simplifies relations (22) and (23) to

$$S_i^{mr} S_{jmr} = A\,(l-1)\,g_{ij}, \tag{33}$$

$$S^{ijk} S_{ijk} = Al\,(l-1)\,. \tag{34}$$

Returning to the triplet T_1, we now expand the associator $[e_\alpha, h_i, h_j]$:

$$(e_\alpha \sigma h_i)\,\sigma h_j - e_\alpha \sigma\,(h_i \sigma h_j) = a h_i \alpha\,(h_j \alpha e_\alpha)\,,$$

which simplifies to

$$\tilde{\alpha}_i \tilde{\alpha}_j - A g_{ij} - \tilde{\alpha}^k S_{ijk} = a \alpha_i \alpha_j . \tag{35}$$

Using relations (31) and (32), contraction of both sides by $\alpha^i \alpha^j$ and by g^{ij} yields, respectively

$$a\,(\alpha,\alpha) + (\tilde{\alpha},\tilde{\alpha}) + A = 0, \tag{36}$$

$$a\,(\alpha,\alpha) - (\tilde{\alpha},\tilde{\alpha}) + Al = 0. \tag{37}$$

These relations can be solved for (α,α) and $(\tilde{\alpha},\tilde{\alpha})$:

$$a\,(\alpha,\alpha) = -\frac{1}{2} A\,(l+1)\,, \tag{38}$$

$$(\tilde{\alpha},\tilde{\alpha}) = \frac{1}{2} A\,(l-1)\,. \tag{39}$$

We see that there are three different equations for (α,α), depending on the value of the composition class parameter a.

Since $l > 0$, relation (38) implies $A = 0$ if $a = 0$,which contradicts the assumption that $A \neq 0$. Hence, the associative case $a = 0$ does not lead to a semi-simple quantal algebra. In the sequel we consider only the cases $a = \pm 1$, so that relation (38) may be more conveniently written as

$$(\alpha,\alpha) = -\frac{a}{2} A\,(l+1)\,. \tag{40}$$

This completes the analysis of the general case $A \neq 0$, $S_{ijk} \neq 0$. We still have to consider the two special cases of vanishing coefficients.

If we take $A = 0$ in relation (17), all calculations remain valid except for the conclusion that S_{ijk} is completely symmetric, for relation (18) no longer implies it. This does not affect the validity of relation (40), which then implies $(\alpha,\alpha) = 0$. Hence, by Theorem 2, there is no non-trivial solution.

If we take $S_{ijk} = 0$ in relation (17), the association identity yields

$$A g_{ij} h_k = A g_{jk} h_i .$$

As an identity in the vectors h_i, it implies $A = 0$, which reduces this case to the previous one. Hence, we conclude that $A \neq 0$ and $S_{ijk} \neq 0$.

As a passing remark, we note that relations (17) and $S_{ijk} = 0$ imply $h_i \sigma h_i = A g_{ij} e$, which is characteristic of Clifford algebras. But since $S_{ijk} = 0$ implies $A = 0$, we conclude that *no non-trivial Clifford algebra is a quantal algebra.*

We now have the necessary relations to prove the key classification theorem for quantal algebras — the counterpart of Theorem 7 for Lie algebras:

Theorem 8 *In a quantal algebra, no string of root vectors can have more than two elements.*

Proof. Let us assume that a string of three elements does exist. Then, there are two root vectors, β and γ, such that, for $\varepsilon \in \{-1, 0, 1\}$, each of the three vectors $\alpha = \beta + \varepsilon\gamma$ is a root vector. Substitution of this expression into relation (40) yields

$$(\beta, \beta) + \varepsilon^2 (\gamma, \gamma) + 2\varepsilon (\beta, \gamma) = -a\frac{1}{2}A(l+1).$$

Since

$$(\beta, \beta) = -a\frac{1}{2}A(l+1)$$

also holds, the difference of these two relations reads

$$\varepsilon^2 (\gamma, \gamma) + 2\varepsilon (\beta, \gamma) = 0.$$

Summing the two equations corresponding to the cases $\varepsilon = \pm 1$ yields $(\gamma, \gamma) = 0$, but, by theorem 2, there are no isotropic root vectors. It follows that ε cannot assume three adjacent values, and, hence, no string can have more than two elements. □

The solutions

According to Theorem 8, the only semi-simple Lie algebras which might support a quantal structure are those with string length not exceeding two. These algebras are the two Cartan families A_l and D_l, and the single algebra B_1. The algebras A_l generate the unitary groups $SU(l)$, i.e., the groups of standard Hilbert space quantum mechanics. With this solution we thus retrieve the standard Hilbert space quantum mechanics, as expected. The algebra B_1 generates the rotation group $SO(3)$, which is locally isomorphic to $SU(2)$. Hence, it is not a strictly non-unitary solution. The algebras D_l generate the orthogonal groups $SO(p, q)$, where $p + q = 2l$. This is the only Cartan family of Lie algebras which may contain non-unitary solutions. We investigate them in the next section.

4. *Non-unitary Quantal Algebras*

If non-unitary algebras exist, we know by Theorem 8 that they are based on Lie algebras from the D_l family. We prove in this section two theorems which further limit the possibilities. The first restricts the rank of the algebra to $l = 3$; the second establishes a relationship between the value of the association parameter a and the sign of the determinant of the metric tensor g_{ij} in the space of root vectors.

Theorem 9 *There are no quantal algebras based on the Lie algebras D_l if $l \neq 3$.*

Proof. The proof consists in showing that the various relations obtained earlier between the root vectors and the coefficients A and S_{ijk} can be satisfied in D_l only if $l = 3$.

To this end, we shall need the explicit expressions for the root vectors in D_l. It is easy to compute from the metric in a $2l$-dimensional space of signature (p, q), where $p + q = 2l$, that they are of the form

$$\alpha_i = \pm r e_a \pm s e_b,$$

where r and s are either 1 or i, and where e_a, e_b, with $a, b \in \{1, \cdots, l\}$ is an arbitrary pair of basis vectors in V^l, i.e., $e_1 = \{1, 0, \cdots, 0\}$, $e_2 = \{0, 1, \cdots, 0\}$, etc. The metric tensor g_{ij} in V^l is diagonal, as it is the restriction of Cartan's metric to the subspace $\mathcal{O}_0$. Its covariant form may be written as

$$g_{ij} = 2 g_i \delta_{ij}, \tag{41}$$

where the system $\{g_1, g_2, ..., g_l\}$ of sign indicators, $g_i = \pm 1$, represents the signature of the metric in V^l. The contravariant form is

$$g^{ij} = \frac{1}{2} g_i \delta^{ij}. \tag{42}$$

We note that while the metric tensor g_{ij} and the root vectors depend on the signature (p, q), the Cartan classification itself does not, which is why the signature is never explicitly considered in texts on classification. With the adjunction of the second product, σ, however, the signature becomes relevant.

To write the general expressions for the root vectors we shall use the square roots of the diagonal elements, which, for convenience, we denote by γ_i,

$$\gamma_i \overset{\text{def}}{=} \sqrt{g_i}, \tag{43}$$

so that

$$\gamma_i^4 = 1. \tag{44}$$

The four root vectors defined by a pair a, b of labels are of the form

$$\alpha_i\left(a, b\right) = \eta\left(\gamma_a \delta_{ai} + \varepsilon \gamma_b \delta_{bi}\right), \tag{45}$$

where $\eta = \pm 1$ and $\varepsilon = \pm 1$. To avoid repetitions, we might take $a < b$, but this is not essential. The labels, taken from the beginning of the alphabet, $a, b, c, ...$, are to be thought of as fixed. The summation convention applies only to the free indices, $i, j, k, ...$ taken from mid-alphabet. Whenever a summation is required over the labels a, b, it is explicitly indicated.

From (42) and (45) we get the contravariant expressions for the root vectors,

$$\alpha^i(a,b) = \frac{1}{2}\eta\left(\gamma_a^3\delta_a^i + \varepsilon\gamma_b^3\delta_b^i\right). \tag{46}$$

The norm,

$$(\alpha,\alpha) \equiv \alpha^i\alpha_i = \frac{1}{2}(\delta_{aa} + \delta_{bb}) = 1, \tag{47}$$

is obviously the same for all root vectors. This is consistent with relation (40), which then implies

$$A(l+1) = -2a. \tag{48}$$

We can now compute the reciprocal root vectors. From relations (31), (46) and (47) follows

$$\begin{aligned}\tilde{\alpha}_k &= \frac{1}{4}\left(\gamma_a^3\delta_a^i + \varepsilon\gamma_b^3\delta_b^i\right)\left(\gamma_a^3\delta_a^j + \varepsilon\gamma_b^3\delta_b^j\right)S_{ijk} \\ &= \frac{1}{4}g_aS_{aak} + \frac{1}{4}g_bS_{bbk} + \frac{1}{2}\varepsilon\gamma_a^3\gamma_b^3S_{abk}.\end{aligned} \tag{49}$$

The orthogonality conditions (30) i.e., $\alpha^k\tilde{\alpha}_k = 0$, imply

$$\begin{aligned}0 &= \left(\gamma_a^3\delta_a^k + \varepsilon\gamma_b^3\delta_b^k\right)\left(g_aS_{aak} + g_bS_{bbk} + 2\varepsilon\gamma_a^3\gamma_b^3S_{abk}\right) \\ &= \left(\gamma_aS_{aaa} + 3\gamma_a^3g_bS_{abb}\right) + \varepsilon\left(\gamma_bS_{bbb} + 3\gamma_b^3g_aS_{aab}\right).\end{aligned}$$

This being valid for $\varepsilon = \pm 1$, both terms in parentheses vanish, leading essentially to the same equation, which, after multiplication by γ_a, simplifies to

$$g_aS_{aaa} + 3g_bS_{abb} = 0 \tag{50}$$

for all $a \neq b$. Thus, for a given label a, this relation represents a set of $l-1$ equations for all labels b different from a. To eliminate this exception, thus allowing b to run over the full index set $\{1,...,l\}$, we adjoin the identity

$$-3g_aS_{aaa} + 3g_aS_{aaa} = 0 \tag{51}$$

to the system of $l-1$ equations (50). The sum of all l equations is then

$$(l-4)\,g_aS_{aaa} + 3\sum_{b=1}^{l} g_bS_{abb} = 0. \tag{52}$$

The summation can be performed using relation (42). It yields

$$\sum_{b=1}^{l} g_bS_{abb} = \frac{1}{2}\sum_{b=1}^{l} g^{bb}S_{abb} = \frac{1}{2}g^{ij}S_{aij} = \frac{1}{2}S_a,$$

which vanishes by relation (32). Hence, relation (52) simplifies to $(l-4)\, S_{aaa} = 0$, implying $l = 4$, or $S_{aaa} = 0$. We consider in turn the cases $l = 2$, $l > 4$, $l = 4$ and $l = 3$.

Rank equal two or greater than four

If $l \neq 4$, then $S_{aaa} = 0$, which with equation (50), implies

$$S_{aaa} = S_{abb} = 0. \tag{53}$$

Thus, all components of S_{ijk} with repeated indices vanish.

Relation (53) simplifies the expression (49) for reciprocal vectors to

$$\tilde{\alpha}_k = \frac{1}{2} \varepsilon \gamma_a^3 \gamma_b^3 S_{abk} \tag{54}$$

with a, b, k all different. For $l = 2$, this is impossible, so that there is no solution for $\tilde{\alpha}_k$. This eliminates rank 2. What follows refers to rank $l > 4$.

The contravariant form of the reciprocal vector is

$$\tilde{\alpha}^k = 2\varepsilon \gamma_a \gamma_b S^{abk}. \tag{55}$$

For its norm, we get

$$(\tilde{\alpha}, \tilde{\alpha}) = S_{abk} S^{abk}, \tag{56}$$

with the reminder that there is a summation over the tensor index k, but not over the labels a, b, so that this relation represents a system of $l\,(l-1)$ scalar equations (all pairs $a \neq b$). Since, by relation (39), all reciprocal root vectors are of the same length, relation (56) implies that $S_{abk} S^{abk}$ has the same value for all labels a, b. Hence, summing this expression over pairs of different labels yields

$$\sum_{a \neq b} S_{abk} S^{abk} = l\,(l-1)\, S_{abk} S^{abk} = l\,(l-1)\, (\tilde{\alpha}, \tilde{\alpha}) \ . \tag{57}$$

On the other hand, by relation (34), the sum on the left hand side is $l\,(l-1)\, A$. Hence,

$$(\tilde{\alpha}, \tilde{\alpha}) = A, \tag{58}$$

and, by relation (39),

$$l = 3, \tag{59}$$

which contradicts the assumption $l > 4$. Thus, from within the infinite family of semi-simple Lie algebras D_l, we have so far eliminated all but D_3 and D_4 as candidates that might support a quantal structure. Next, we eliminate D_4.

Rank four

For $l = 4$, equations (48) and (39) yield, respectively,

$$A = -\frac{2}{5}a, \tag{60}$$

$$(\tilde{\alpha}, \tilde{\alpha}) = -\frac{3}{5}\,. \tag{61}$$

We next reduce the coefficients S_{ijk} to two vectors, Z_a, U_a, defined as

$$Z_a \stackrel{\text{def}}{=} S_{aaa}, \tag{62}$$

$$U_a \stackrel{\text{def}}{=} S_{bcd}, \tag{63}$$

where a, b, c, d are all different, i.e., they represent any permutation of $1, 2, 3, 4$. Then, by relation (50),

$$S_{abb} = -\frac{1}{3}g_a g_b Z_a. \tag{64}$$

Computing $(\tilde{\alpha}, \tilde{\alpha})$ from the expression (49), we get

$$\tilde{\alpha}_k \tilde{\alpha}^k = \frac{1}{16}\left[S_{aak}S^k_{aa} + S_{bbk}S^k_{bb} + 2g_a g_b S_{aak}S^k_{bb} + 4g_a g_b S_{abk}S^k_{ab}\right]$$

$$+\frac{\varepsilon}{8}\left[\gamma_a\gamma_b^3 S_{aak}S^k_{ab} + \gamma_a^3\gamma_b S_{bbk}S^k_{ab} + \gamma_a\gamma_b^3 S_{abk}S^k_{aa} + \gamma_a^3\gamma_b S_{abk}S^k_{bb}\right].$$

As this represents two equation (for $\varepsilon = \pm 1$ respectively), taking their difference implies that the second term vanishes. Together with equation (61), the first term then yields

$$S_{aak}S^k_{aa} + S_{bbk}S^k_{bb} + 2g_a g_b S_{aak}S^k_{bb} + 4g_a g_b S_{abk}S^k_{ab} = -\frac{48}{5}. \tag{65}$$

We now expand each term using relation (42). For example

$$\begin{aligned} S_{aak}S^k_{aa} &= \frac{1}{2}\sum g_k S_{aak}S_{aak} \\ &= \frac{1}{2}\left[g_a S_{aaa}S_{aaa} + g_b S_{aab}S_{aab} + g_c S_{aac}S_{aac} + g_d S_{aad}S_{aad}\right]. \end{aligned}$$

In the shorter notations defined by relations (62), (63), (64), equation (65) reads

$$\frac{1}{9}\left[2g_a Z_a^2 + 2g_b Z_b^2 + g_c Z_c^2 + g_d Z_d^2\right] + g\left[g_c U_c^2 + g_d U_d^2\right] + \frac{24}{5} = 0, \tag{66}$$

where

$$g \overset{\text{def}}{=} g_a g_b g_c g_d = \det(g_{ij}) \,.$$

Relation (66) represents a system of 4! equations (the permutations of 4 labels), but due to the symmetry in the label pairs a, b and c, d, only 6 of the these equations are independent. As there are 8 unknowns (Z_1, U_1 to Z_4, U_4) we need additional equations to compute these unknowns.

We obtain another system of independent equations from relation (33), which, for $l = 4$, reads

$$S_i^{rs} S_{jrs} = -\frac{6}{5} g_{ij}.$$

Taking $i = j = a$ and expanding the sum as in the previous case, one obtains

$$\begin{aligned} 0 \quad = \quad & \frac{1}{9}\left[6 g_a Z_a^2 + g_b Z_b^2 + g_c Z_c^2 + g_d Z_d^2\right] \\ & + g\left[g_b U_b^2 + g_c U_c^2 + g_d U_d^2\right] + \frac{24}{5}. \end{aligned} \tag{67}$$

This represents a system of 4 equations, which, with the system (66), yields 10 equations for 8 unknowns. In principle, this system is overdetermined. To verify that it actually is, we take the difference of the relations (66) and (67):

$$\frac{4}{9} g_a Z_a^2 - \frac{1}{9} g_b Z_b^2 + g g_b U_b^2 = 0. \tag{68}$$

Adding to this relation the relation

$$\frac{4}{9} g_b Z_b^2 - \frac{1}{9} g_a Z_a^2 + g g_a U_a^2 = 0$$

obtained by interchanging a and b, one obtains

$$\left[\frac{1}{3} g_a Z_a^2 + g g_a U_a^2\right] = -\left[\frac{1}{3} g_b Z_b^2 + g g_b U_b^2\right].$$

For this to be valid for all six pairs of labels, both sides must vanish, implying

$$g g_a U_a^2 = -\frac{1}{3} g_a Z_a^2.$$

Substitution of this result into relation (68) yields $g_a Z_a^2 = g_b Z_b^2$. Hence, there exists a constant K such that

$$\begin{aligned} g g_a Z_a^2 \quad &= \quad K, \\ g g_a U_a^2 \quad &= \quad -\frac{1}{3} K, \end{aligned}$$

for $a = 1, 2, 3$ or 4. Substitution of these results into equation (67) leads to the absurd conclusion $\frac{24}{5} = 0$. Hence, no non-unitary quantal algebra exists for $l = 4$. This leaves $l = 3$ as the only possibility. □

We next turn to the question of signature for the case $l = 3$.

Rank three

Having eliminated all dimensions other than $l = 3$, we still have no proof that a quantal algebra based on D_3 actually exists, but, if it does, its metric is subject to the following theorem:

Theorem 10 *If a quantal algebra based on D_3 exists, the determinant of the metric in V^3 is defined by the association parameter a:*

$$\det(g_{ij}) = -a. \tag{69}$$

Proof. The algebra D_3 has four signature variants with respect to the metric in V^3, namely $(-,-,-)$, $(-,-,+)$, $(-,+,+)$, $(+,+,+)$. They are represented by the sign indicators (g_1, g_2, g_3).

Let S denote the only non-vanishing component of the symmetric tensor S_{ijk}, i.e.,

$$S = S_{123}. \tag{70}$$

Its contravariant form is

$$S^{123} = S_{123} g^{11} g^{22} g^{33} = \frac{1}{8} S g_1 g_2 g_3 = \frac{1}{8} SD, \tag{71}$$

where

$$D = \det(g_{ij}) = g_1 g_2 g_3. \tag{72}$$

Hence,

$$S_{ijk} S^{ijk} = \sum_P S_{123} S^{123} = \frac{3!}{8} S^2 D, \tag{73}$$

where the summation is performed over all permutations of the set of indices $1, 2, 3$. For $l = 3$, relations (48) and (34) yield, respectively,

$$A = -\frac{a}{2}, \tag{74}$$

$$S_{ijk} S^{ijk} = -3a. \tag{75}$$

Hence, from relation (73) follows $S^2 D = -4a$. Writing $D = \Delta^2$, where

$$\Delta \stackrel{\text{def}}{=} \gamma_1 \gamma_2 \gamma_3, \tag{76}$$

and taking into account that $D^2 = 1$, one obtains the following expression for S,

$$S = 2i\Theta\sqrt{a}\Delta, \tag{77}$$

where we have denoted by Θ the sign of the square root, i.e., $\Theta = \pm 1$. As it characterizes the quantal algebra, we introduce a name for it:

Definition 5 *The* ***orientation*** *of a quantal algebra is the sign Θ of the square root of the associativity constant a.*

The final expressions for the sigma products in the Cartan subalgebra $\mathcal{O}_0$ follow from relations (17), (41), (42) and (74) as

$$h_i \sigma h_i = -a g_i e, \tag{78}$$

$$h_i \sigma h_j = i\Theta\sqrt{a}\Delta g_k h_k \tag{79}$$

where i, j, and k are all different. Since $h_i \sigma h_j$ is a real observable, the coefficient $i\Theta\sqrt{a}\Delta g_k$ must be real, i.e., $i\sqrt{a}\Delta$ must be real. There are two possibilities:

Case of $a = 1$: In this case, $\Delta = \pm i$, and, hence, $D = -1$. This corresponds to the two signatures $(-,-,-)$ and $(-,+,+)$.

Case of $a = -1$: In this case, $\Delta = \pm 1$, and, hence, $D = 1$. This corresponds to the two signatures $(+,+,+)$ and $(-,-,+)$.

These results can also be verified directly. To this effect, we observe that the abstract subalgebra $\{\mathcal{O}_0 \oplus e\mathbb{R}, \sigma\}$ is associative (by the association relation and the fact that the alpha products on $\mathcal{O}_0$ vanish). Taking $i\sqrt{a}\Delta = 1$, the multiplication rules (78), (79) read

$$h_i \sigma h_i = -a g_i e, \tag{80}$$

$$h_i \sigma h_j = \Theta g_k h_k. \tag{81}$$

Considering the coefficients g_j to be unknown, but requiring associativity,

$$(h_i \sigma h_i)\, \sigma h_j = h_i \sigma\, (h_i \sigma h_j)\,,$$

one obtains $-a g_i h_j = \Theta^2 g_j g_k h_j$, which implies $|g_i| = 1$ and $D = g_1 g_2 g_3 = -a$. $\Box$

This completes the proof that a concrete non-unitary quantal algebra exists. It has four variants, distinguished by the four possible systems of sign indicators of the metric in the space V^3 of root vectors. Two correspond to $a = 1$, and two to $a = -1$.

We still have to derive the explicit multiplication tables for the products sigma and alpha. This can be done in terms of root vectors and eigenvectors in the mathematical style of the present paper, but the calculations are tedious, while the results thus obtained are not in a form adapted to further work. By contrast, the calculations are quite straightforward in the formalism of linear algebra (Part III) — which formalism we shall also need as an intermediate step in the construction of the quantionic algebra.

5. *Conclusion*

Since Parts I and II of the present work make very little reference to physics, their physical relevance might not yet be evident. Referring to the history of mathematics, which in great measure influenced our approach to the study of foundations in physics, the following discussion should help put the approach in question in perspective and justify it heuristically.

Mathematical thinking can be classified, introducing an *ad hoc* terminology, as *existential* or *technical.* Examples of the former are Gauss's fundamental theorem of algebra, which guarantees the *existence* of a complex root for every polynomial with complex coefficients (1799), and Galois' proof of the *non-existence* (in the general case) of algebraic expressions for the roots if the polynomial is of degree greater than four (1832). By contrast, the algebraic computation of these roots — the Babylonian solution for the quadratic equation (1500 BC), and Cardano's solution for the cubic equation (1545) — are *technical.* As the dates suggest, technical mathematical thinking, bucking logic, antedates concerns about existential questions. Making allowance for exceptions, like Euclid's proof of the *non-existence* of a rational expression for the square root of two, technical thinking has been dominant in mathematics until relatively recently — with consequences most wasteful of mental effort, as exemplified by two millennia of futile attempts at "squaring the circle", until Lindeman's proof of the tanscendence of π (1882) established the *non-existence* of a solution.

Similarly in physics, the application of symmetry arguments and related conservation laws eliminates in bulk infinite sets of candidate solutions to a problem by proving their *non-existence* — leaving for analysis by *technical* computations a much smaller set that escapes elimination. Advances in modern theoretical physics stem from a systematic interplay of the two modalities of thinking.

The work completed so far in Parts I and II of the present paper is strictly of existential type. If it "feels like mathematics", not like physics, it is because the problem we are considering is not *within* physical theories, but *about* them. We shall briefly review our objective and approach by making three observations: (1) The existence of a maximal velocity c is structurally accounted for by relativity. (2) The existence of a minimal action, $\hbar$, is structurally accounted for by quantum mechanics. (3) The existence of both simultaneously is not yet structurally accounted for by any theory. Our objective is to find out if such a theory is possible. To this end, the existential vs. technical classification yields its first fruits by suggesting a methodology. Let us consider both possibilities.

The technical approach: Attempting to *construct* a unifying theory from relativity and quantum mechanics entails modifying at least the latter by whatever means come to mind. This has undoubtedly been attempted countless times during the last seventy years. While some of these attempts have been publishable, none has been successful — which strongly suggests that it cannot be done. As Steven Weinberg points out, Hilbert space is "too rigid" to admit any modifications that would not destroy what is essential in it for quantum mechanical interpretation (at least for modifications within currently available mathematics).

The existential approach: Surprisingly, this approach has apparently not yet been systematically explored. Our methodology consists of two steps:

Step 1: Extract from quantum mechanics some very general features considered essential, and then work them into an abstract mathematical structure. This ensures that they form a self-consistent mathematical object. This step has been completed in Part I, the generated structure being quantal algebra. We note that it involves no modifications of existing concepts. The thinking is not even goal-oriented, as *we are not trying to achieve anything in particular.* We are only hand-picking, on heuristic grounds, some concepts out of quantum mechanics and ensuring their mutual consistency in an abstract setting, at which point the essential part of the work is completed. The abstract structure in question, the quantal algebra, happens to be so rigid that it "forces the non-existence" of almost all imaginable realizations, allowing only a few exceptions.

Step 2: Find the exceptions, i.e., all realizations of the abstract quantal algebra. The procedure, completed in the present part of the work, extends Cartan's classification of semi-simple Lie algebras, as illustrated in Figure 1.

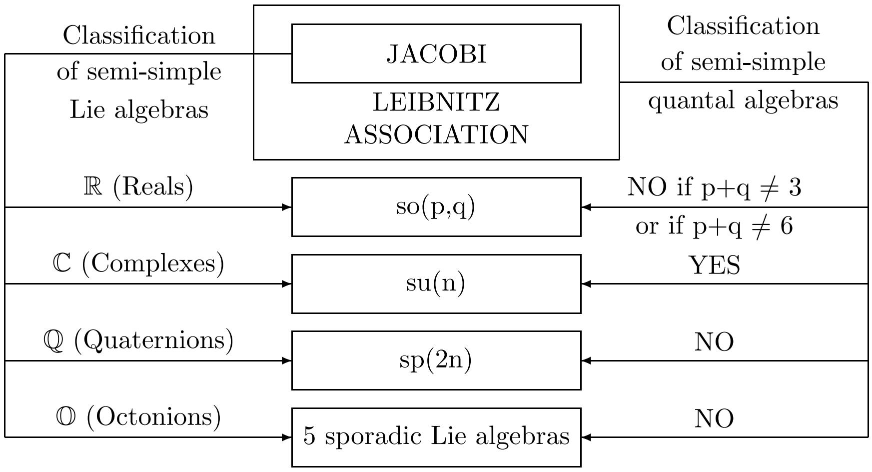

Fig. 1. Realizations of the Semi-simple Structures

The left side shows Cartan's classification, which stems exclusively from the structure of the Lie algebra — essentially the Jacobi identity. There are three infinite families of solutions (orthogonal, unitary and symplectic) and a set of five sporadic solutions. They correspond respectively, to orthogonal groups in sesquilinear real, complex, quaternionic and octonionic spaces. A quantal algebra being a Lie algebra with an additional product and two additional identities (Leibnitz and association), the number of possible realizations is smaller. The family of unitary groups is preserved (expectably so, as it is the underlying structure of standard quantum mechanics), but of the other Cartan solutions, only the rotations in real spaces of 3 and 6 dimensions are not eliminated. The signatures are arbitrary.

We see that step (2) is also devoid of technical arguments. It is purely existential. Thus, if a generalization of quantum mechanics exists (and it is needed for unification with relativity), it is somehow based on one of the Lie algebras $so(p,q)$, where $p+q=3$ or $p+q=6$. These results were not *constructed*. They merely *withstood all attempts at proving that they don't exist*. This concludes the essentially existential mathematics of the present work. Parts III and IV will be *technically* dedicated to the quantionic algebra — a relativistic number system based on the quantal algebra over $so\,(2,4)$.

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SUŠTINSKI RELATIVISTIČKA KVANTNA TEORIJA Dio II. SVRSTAVANJE RJEŠENJA

Apstraktna kvantalna algebra razvijena u Dijelu I ovog rada opisuje zajedničku strukturu dviju poznatih mehanika, klasične i kvantne. Sama ta algebra nije fizika. Ona je matematički sustav, ili, kako bi neki mogli reći samo matematika, točan prigovor ako bi kvantalna algebra bila sama sebi ciljem, jer fizika nije u apstraktnim teorijama već u njihovim stvarnim realizacijama. Stoga se pitamo, postoji li bar jedna stvarna realizacija kvantalne algebre, jer medu tima mogle bi se naći generalizacije kvantne mehanike koje vrijede u fizici. Potraga za realizacijama apstraktne teorije je poznata pod nazivom *strukturna teorija* ili *klasifikacijski problem*. To je obično vrlo težak zadatak, no u ovom je slučaju relativno lagan jer su osnove već postavljene Cartanovom klasifikacijom polujednostavnih Lievih algebri. Budući da kvantalna algebra sadrži jednu Lievu algebru, trebamo samo primijeniti standardne rezultate postavljanjem dodatnih uvjeta. Ishod je toga da polujednostavna kvantalna algebra ima točno dvije realizacije. Izraženo preko teorije grupa, jedna je beskonačna familija unitarnih grupa, $SO\,(n)$, (tj., standardna kvantna mehanika), a druga je posebno rješenje, $SO\,(2,4)$. Klasična mehanika nije rješenje jer zahtjev polujednostavnosti uklanja kanonsku grupu. Stoga, ako se kvantna mehanika može generalizirati. ta je generalizacija na neki način u svezi s grupom $SO\,(2,4)$. Ta grupa sadrži relativističku strukturu prostora–vremena, pa se čini da je moguća suštinski kovarijantna generalizacija.