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DIRAC EQUATION WITH FRACTIONAL DERIVATIVES OF ORDER 2/3

ANDREA RASPINI

Department of Physics, SUNY at Fredonia, Fredonia, NY14063, U. S. A.
E-mail: Raspini@Fredonia.EDU

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In conventional spacetime, a Dirac-like equation with fractional derivatives of order 2/3 is introduced. The corresponding γ matrix algebra relates to generalized Clifford algebras: finite representations exist with smallest dimension $N = 9$.

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1. Introduction

This is a short paper which introduces a Dirac-like equation with fractional derivatives of the order 2/3. Fractional calculus [1–3] is briefly reviewed in Sect. 2 and the aforementioned equation is presented in Sect. 3, as a cube root of the Klein-Gordon equation [4]. The resulting γ matrix algebra (Sect. 4) is of the cubic polynomial Clifford type [5,6], which relates to the corresponding generalized Clifford algebras [7,8].

The notation is rather standard. In particular, the summation convention is applied to repeated up and down labels, and units are such that $\hbar = c = 1$. All Greek indices run through the values $0, 1, 2, 3$, while the Latin index ℓ takes on the values $1, 2, 3$; other labels are defined as needed. An attempt is made at distinguishing superscripts from powers: for instance, $(m)^2$ and $(\not{Q})^3$ are powers, while x^3 indicates a specific variable with superscript 3. Non-integral powers of positive numerical quantities are conveniently taken in their real positive branch, and pseudoeuclidean squares of four-vectors are identified like this: $\mathrm{P}^{[2]}$. The curly bracket notation is used for ordered sets: for example, $\{x^\lambda\}$ indicates four objects in the order $0-3$.

2. *Review of fractional calculus*

Fractional calculus can be introduced in several different ways [1,2]. One of the most evident formulations is based on the fundamental definition (Liouville)

$$\mathrm{D}^f G(u) = \frac{1}{[-(f+1)]!} \int\limits_{-\infty}^{u} (u-\xi)^{-(f+1)}\, G(\xi)\, \mathrm{d}\xi \qquad (\text{for } f<0), \tag{1}$$

and on the auxiliary one

$$\mathrm{D}^f G(u) = \mathrm{D}^F [\mathrm{D}^{(f-F)} G(u)] \qquad (\text{for } f \geq 0), \tag{2}$$

where F is the smallest positive integer exceeding f, and D^F is an ordinary differentiation of the F-th order with respect to the variable u.

The linear operation D^f is called fractional differentiation[1] of order f relative to the variable u (or, especially for negative f, fractional integration of order $\bar{f} = -f$). Here, f, u and ξ are all real (but generalizations to the complex domain are possible), and the function $G(u)$ may be real or complex valued. For functions $G(u)$ which are well-behaved and strongly vanishing (along with their ordinary derivatives of all orders) at $-\infty$, the indicated fractional derivatives exist and have some interesting properties [1]:

(i) if f is a non-negative integer, $\mathrm{D}^f G(u)$ reduces to the ordinary derivative of order f relative to the variable u [including $\mathrm{D}^0 G(u) = G(u)$ for the zeroth order derivative];

(ii) if $\bar{f}$ is a positive integer, $\mathrm{D}^f G(u)$ reduces to an ordinary $\bar{f}$-fold integration of the function G;

(iii) the law of exponents $\mathrm{D}^{f_2}[\mathrm{D}^{f_1} G(u)] = \mathrm{D}^{f_1}[\mathrm{D}^{f_2} G(u)] = \mathrm{D}^{(f_1+f_2)} G(u)$ is valid for all (real) values of f_1 and f_2.

Partial fractional derivatives may be introduced as a straightforward generalization of the outlined formalism. For instance, if $x = \{x^\lambda\}$ are real spacetime variables and $H = H(x)$ is a function of them, the derivative $\mathrm{D}^f_\mu H(x)$ indicates a differentiation of order f with respect to the variable x^μ, treating the remaining variables as "spectators". Under suitable assumptions for $H(x)$, properties like those in (i)–(iii) can be established for differentiations relative to each specific variable. Furthermore, for mixed derivatives [2]

$$\mathrm{D}^{f_2}_\nu [\mathrm{D}^{f_1}_\mu H(x)] = \mathrm{D}^{f_1}_\mu [\mathrm{D}^{f_2}_\nu H(x)]\,. \tag{3}$$

In concluding this section, it is important to re-iterate that some of the listed properties are valid under sufficiently restrictive conditions for the functions involved: if these conditions are relaxed, the same properties may not be applicable with generality (e.g., the law of exponents). For specific details, see, for instance Refs. [1,3].

[1] Note that the various existing formulations of fractional calculus are not necessarily all equivalent, as they may lead to slightly different types of fractional differentiations [1].

A remark about notation. As previously defined, let D^f indicate the fractional differentiation of order f relative to the variable u. Some authors introduce the symbol [1]:

$$G^{(f)}(u) = \mathrm{D}^f G(u) \,. \tag{4}$$

If changes of variables are contemplated, it is important to keep in mind the intended meanings of these two types of notation. For example, $\mathrm{D}^f G(-u)$ stands for $\mathrm{D}^f \mathtt{G}(u)$, with $\mathtt{G}(u) = G(-u)$. On the other hand, $G^{(f)}(-u)$ implies that $\mathrm{D}^f G(u)$ is calculated first, and then $u \to -u$. The analogy with the notations employed for ordinary derivatives is evident.

3. *Dirac equation*

In a frame of reference $\mathcal{X}$ of real spacetime coordinates $x = \{x^\lambda\}$ and pseudoeuclidean metric $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1)$, the massive Dirac equation may be written as follows [4]

$$\mathrm{i}\Gamma^\alpha \partial_\alpha \Phi(x) = m\Phi(x) \,, \qquad m > 0 \,. \tag{5}$$

Here, $\Phi(x)$ is a complex four-spinor and the 4×4 Dirac matrices Γ^λ (in a fixed chosen representation) obey the usual rules

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2g^{\mu\nu} I \,, \qquad (\Gamma^\mu)^\dagger = \Gamma^0 \Gamma^\mu \Gamma^0 \,, \tag{6}$$

where I is the 4×4 identity matrix, and $\partial_\mu = \mathrm{D}^1_\mu$ in the notation of the previous section.

With $\mathrm{P}_\mu = \mathrm{i}\partial_\mu$ and with the squared four-momentum operator defined as

$$\mathrm{P}^{[2]} \equiv \mathrm{P}_\alpha g^{\alpha\beta} \mathrm{P}_\beta = -\partial_\alpha g^{\alpha\beta} \partial_\beta \,, \tag{7}$$

it is well known that the differential operator at the left-hand side of Eq. (5)

$$\not{\mathrm{P}} = \mathrm{i}\Gamma^\alpha \partial_\alpha \tag{8}$$

satisfies

$$(\not{\mathrm{P}})^2 \equiv \not{\mathrm{P}}\not{\mathrm{P}} = \mathrm{P}^{[2]} \,, \tag{9}$$

on well-behaved spinorial functions[2]. In other words, $\not{\mathrm{P}}$ is a "square root" of $\mathrm{P}^{[2]}$ and this is a crucial issue in the Dirac theory: all (well-behaved) solutions $\Phi(x)$ of Eq. (5) are eigenstates of $\mathrm{P}^{[2]}$ corresponding to the eigenvalue $(m)^2$.

[2] Sufficient condition: the continuity of all second partial derivatives of each spinorial component.

It is interesting to examine whether meaningful modifications of the Dirac equation exist, in terms of fractional derivatives. Specifically, this paper considers the case

$$-\gamma^{\alpha}\mathrm{D}_{\alpha}^{2/3}\,\Psi(x) = (m)^{2/3}\,\Psi(x)\,, \tag{10}$$

where the following condition is imposed:

$$(\not{Q})^3 \equiv \not{Q}\not{Q}\not{Q} = \mathrm{P}^{[2]}\,, \qquad \text{with } \not{Q} = -\gamma^{\alpha}\mathrm{D}_{\alpha}^{2/3}\,, \tag{11}$$

to be operated on sufficiently well-behaved spinorial functions (i.e., as to allow the validity of the relevant properties outlined in Sect. 2). Here, the (complex) matrices γ^{λ} are $N \times N$ and $\Psi(x)$ is a complex N-spinor (N to be determined). With Eq. (11) enforced, the fractional differential operator $\not{Q}$ is a "cube root" of $\mathrm{P}^{[2]}$ and all (well-behaved) solutions $\Psi(x)$ of Eq. (10) are eigenstates of $\mathrm{P}^{[2]}$ with eigenvalue $(m)^2$.

In terms of ordinary calculus, Eq. (10) is an integro-differential equation, but the language and notation of fractional calculus appear to be more convenient and suggestive in the present context. It is also pointed out that the terms "spinor" and "spinorial", as used in relationship to Eq. (10), are meant to refer to the column structure of $\Psi(x)$, not to the validity of conventional spinorial transformations under changes of coordinates. As a matter of fact, the transformation properties of these generalized spinors $\Psi(x)$ are yet to be determined.

4. The γ matrices

An examination of Eq. (11), together with the results of Sect. 2, shows that the matrices $\gamma = \{\gamma^{\lambda}\}$ should satisfy

$$[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}]_{+} = e^{\mu\nu\rho} I\,, \tag{12}$$

where I is now the $N \times N$ identity matrix. Here, the bracket denotes symmetrization over the indices

$$[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}]_{+} = \frac{1}{3!}\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho} + \gamma^{\mu}\gamma^{\rho}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}\gamma^{\rho} + \gamma^{\nu}\gamma^{\rho}\gamma^{\mu} + \gamma^{\rho}\gamma^{\mu}\gamma^{\nu} + \gamma^{\rho}\gamma^{\nu}\gamma^{\mu}\right), \tag{13}$$

and the symbol $e^{\mu\nu\rho}$ is so defined

$$e^{\mu\nu\rho} = 0 \;\text{ unless }\; \mu = \nu = \rho\,, \tag{14}$$

$$e^{000} = 1 = -\,e^{\ell\ell\ell}\,. \tag{15}$$

A slight simplification is obtained by setting $\gamma_{\sigma} = g_{\sigma\alpha}\gamma^{\alpha}$ which gives

$$[\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}]_{+} = \delta_{\mu\nu\rho}\, I\,, \tag{16}$$

where $\delta_{\mu\nu\rho}$ is the Kronecker symbol.

Equation (16) and similar ones have been approached in the context of three main related topics: linearization of polynomials [5], polynomial Clifford algebras [6], and generalized Clifford algebras [7,8]. The author is grateful to Dr. Antonio Lopez Almorox (Department of Mathematics, University of Salamanca, Salamanca, Spain) for pointing out these references and many others, and for useful discussions and explanations. It is known that Eq. (16) can be solved, and it appears [5] that $N = 9$ is the smallest dimension.

5. *Conclusions*

Future papers will examine the algebraic aspects [6,5,8] of Eq. (16) in more detail, and attempt to find simple solutions for the generalized spinors of Eq. (10). It is hoped that appropriate meanings and interpretations will follow, if the subject is deemed to have sufficient interest and validity (e.g., in the context of mean field theory or similar). In the meantime, it is observed that fractional calculus is linked to fractal theory [9], and it is the author's speculation that it might also relate to dimensional regularization [10].

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DIRACOVA JEDNADŽBA S RAZLOMČANIM DERIVACIJAMA REDA 2/3

Uvodi se Diracova jednadžba s razlomčanim derivacijama reda 2/3 u konvencijskom prostoru. Odgovarajuća γ-matrična algebra je srodna poopćenim Cliffordovim algebrama: postoje reprezentacije dimenzije $N = 9$.