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THE EXTENDED JACOBIAN ELLIPTIC FUNCTION EXPANSION METHOD AND ITS APPLICATION TO NONLINEAR WAVE EQUATIONS

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In this work an extended Jacobian elliptic function expansion method is applied to construct the exact periodic solutions of two nonlinear wave equations. The periodic solutions obtained by this method can be reduced to the solitary wave solutions under certain limiting conditions.

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1. Introduction

Studying nonlinear problems is an important subject in every scientific field, including social science field. These problems are usually characterized by nonlinear evolution partial differential equations (NLEPDEs). The construction of exact solutions of the associated nonlinear equations plays an important role in understanding the nonlinear problems. Recently, many methods have been proposed, such as the homogeneous balance method [1–3], the hyperbolic tangent expansion method [4–5], the trial function method [6], the nonlinear transformation method [7] and sine-cosine method [8]. Many exact solutions have been obtained. However, these methods can only get the shock and solitary wave solutions and cannot obtain the periodic solutions. Liu et al. have proposed the Jacobian elliptic function expansion method [9,10] and obtained many periodic solutions. In this paper, we will use the extended Jacobian elliptic function expansion method to construct new exact periodic solutions of the Benjamin-Bona-Mahoni (BBM) equation and the nonlinear Klein-Gordon equation. Under limiting conditions, the periodic solutions can be reduced to the corresponding solitary wave solutions.

2. The extended Jacobi elliptic function expansion method

Consider a given nonlinear wave equation

$$N\left(u, u_t, u_x, u_{xx}, u_{tt}, \cdots\right) \quad = \quad 0. \tag{1}$$

We seek its traveling wave solutions in the form

$$u(x,t) = u(\xi), \qquad \xi = k(x - ct), \tag{2}$$

where k and kc are the wave number and phase velocity, respectively. Substituting (2) into Eq. (1) yields a nonlinear ordinary differential equation (ODE)

$$F\left(u, \frac{\mathrm{d}u}{\mathrm{d}\xi}, \frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}, \cdots\right) \quad = \quad 0. \tag{3}$$

Let us define the concept of "rank". If the nonlinear term in the above reduced ODE can be written as

$$u^{k_0} u'^{k_1} (u'')^{k_2} \cdots (u^{(m)})^{k_m}, \tag{4}$$

with k_i real constants, then the rank of this term is defined as the number

$$0k_0 + k_1 + 2k_2 + \cdots + mk_m \tag{5}$$

that is, by the sum of the number of $\mathrm{d}/\mathrm{d}\xi$. If the rank of every term in the reduced ODE is even or odd, we can use following extended Jacobian elliptic function expansion method.

We assume that Eq. (3) has the solutions in the form

$$u(\xi) \quad = \quad \sum_{j=0}^{n} a_j f_i^j + \sum_{j=1}^{n} b_j f_i^{j-1} g_i, \tag{6}$$

where f_i and g_i are the following pairs of the closed Jacobian elliptic functions

$$\begin{aligned} f_1(\xi) &= \mathrm{sn}(\xi), \qquad g_1(\xi) = \mathrm{cn}(\xi), \\ f_2(\xi) &= \mathrm{sn}(\xi), \qquad g_2(\xi) = \mathrm{dn}(\xi), \\ f_3(\xi) = \mathrm{ns}(\xi) &= \frac{1}{\mathrm{sn}(\xi)}, \qquad g_3(\xi) = \mathrm{cs}(\xi) = \frac{\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)}, \end{aligned} \tag{7}$$

where $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ and $\mathrm{dn}(\xi)$ are the Jacobian elliptic sine function, the Jacobian elliptic cosine function and the Jacobian elliptic function of the third kind, respectively. They have the following relations

$$\mathrm{cn}^2(\xi) = 1 - \mathrm{sn}^2(\xi), \quad \mathrm{dn}^2(\xi) = 1 - m^2\mathrm{sn}^2(\xi), \quad \mathrm{cs}^2(\xi) = \mathrm{ns}^2(\xi) - 1, \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{sn}(\xi) = \mathrm{cn}(\xi)\,\mathrm{dn}(\xi), \quad \frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{cn}(\xi) = -\mathrm{sn}(\xi)\,\mathrm{dn}(\xi), \quad \frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{dn}(\xi) = -m^2\mathrm{sn}(\xi)\,\mathrm{cn}(\xi) \tag{9}$$

with the modulus m ($0 < m < 1$). To determine n, we can use the homogeneous balance method, which balances the degree of the highest order linear term with the nonlinear term [9,10]. In addition when the modulus $m \to 1$, then $\mathrm{sn}(\xi) \to \tanh(\xi)$, $\mathrm{sn}(\xi) \to \mathrm{sech}(\xi)$ and $\mathrm{dn}(\xi) \to \mathrm{sech}(\xi)$.

3. *Modified BBM equation*

We consider now the Benjamin-Bona-Mahoni (BBM) equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \alpha u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial t \partial u^2} = 0\,. \tag{10}$$

Substituting (2) into Eq. (10) yields

$$(1-c)\frac{\mathrm{d}u}{\mathrm{d}\xi} + \alpha u^2 \frac{\mathrm{d}u}{\mathrm{d}\xi} - k^2 c\beta \frac{\mathrm{d}^3 u}{\mathrm{d}\xi^3} = 0. \tag{11}$$

3.1. Jacobian elliptic functions $\mathrm{sn}(\xi)$ *and* $\mathrm{cn}(\xi)$ *expansion*

We assume that Eq. (11) has the solution

$$u(\xi) = \sum_{j=0}^{n} a_j \mathrm{sn}^j(\xi) + \sum_{j=1}^{n} b_j \mathrm{sn}^{j-1}(\xi)\,\mathrm{cn}(\xi). \tag{12}$$

The highest degree of (12) is

$$O\left(u(\xi)\right) = n, \tag{13}$$

and we can also get

$$O(\frac{\mathrm{d}u}{\mathrm{d}\xi}) = n+1, \qquad O(u^2\frac{\mathrm{d}u}{\mathrm{d}\xi}) = 3n+1, \qquad O(\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}) = n+3. \tag{14}$$

Balancing the highest order of derivative term and nonlinear term in Eq. (11), we can obtain

$$n = 1 \tag{15}$$

So the solution of Eq. (11) can be assumed as

$$u(\xi) = a_0 + a_1 \mathrm{sn}(\xi) + b_1 \mathrm{cn}(\xi). \tag{16}$$

Substituting (16) into Eq. (11) yields

$$\begin{aligned}&\big[(1-c)+\alpha(a_0^2+b_1^2)+k^2c\beta\big(1+m^2\big)\big]a_1\cdot\mathrm{cn}(\xi)\mathrm{dn}(\xi)\\&+\big[-(1-c)b_1-\alpha\big(a_0^2+b_1^2\big)b_1-k^2c\beta\big(1+4m^2\big)b_1+2\alpha a_1^2\big]\cdot\mathrm{sn}(\xi)\mathrm{dn}(\xi)\\&+2\alpha a_0\big(a_1^2-b_1^2\big)\cdot\mathrm{sn}(\xi)\mathrm{cn}(\xi)\mathrm{dn}(\xi)\\&+\big[-2\alpha b_1+\alpha\big(a_1^2-b_1^2\big)-6k^2c\beta m^2\big]a_1\cdot\mathrm{sn}^2(\xi)\mathrm{cn}(\xi)\mathrm{dn}(\xi)\\&+\big[-2\alpha a_1^2-\alpha\big(a_1^2-b_1^2\big)b_1+6k^2c\beta m^2b_1\big]\cdot\mathrm{sn}^3(\xi)\mathrm{dn}(\xi)=0\end{aligned}\tag{17}$$

from which it is determined that

$$a_0=0,\qquad a_1=0,\qquad b_1=\pm\sqrt{\frac{-6k^2c\beta m^2}{\alpha}},\qquad k^2=\frac{c-1}{c\beta(1-2m^2)},\tag{18}$$

or

$$a_0=0,\qquad b_1=0,\qquad a_1=\pm\sqrt{\frac{6k^2c\beta m^2}{\alpha}},\qquad k^2=\frac{c-1}{c\beta(1+m^2)},\tag{19}$$

or

$$a_0=0,\qquad b_1=\pm\mathrm{i}a_1=\pm\sqrt{\frac{3m^2(c-1)}{\alpha(2-m^2)}},\qquad k^2=\frac{2(c-1)}{c\beta(2-m^2)}.\tag{20}$$

Thus we can obtain the periodic solutions

$$u_1=\pm\sqrt{\frac{6m^2(c-1)}{\alpha(1+m^2)}}\cdot\mathrm{sn}\bigg(\sqrt{\frac{c-1}{c\beta(1+m^2)}}(x-ct)\bigg),\tag{21}$$

$$u_2=\pm\sqrt{\frac{6m^2(c-1)}{\alpha(2m^2-1)}}\cdot\mathrm{cn}\bigg(\sqrt{\frac{c-1}{c\beta(1-2m^2)}}(x-ct)\bigg),\tag{22}$$

$$\begin{aligned}u_3=&\pm\sqrt{\frac{6m^2(c-1)}{\alpha(2m^2-1)}}\bigg[\mathrm{cn}\bigg(\sqrt{\frac{2(c-1)}{c\beta(2-m^2)}}(x-ct)\bigg),\\&\pm\mathrm{i}\cdot\mathrm{sn}\bigg(\sqrt{\frac{2(c-1)}{c\beta(2-m^2)}}(x-ct)\bigg)\bigg].\end{aligned}\tag{23}$$

When the modulus $m\to1$, Eqs. (21)–(23) can be reduced to following solitary solutions

$$u_1'=\pm\sqrt{\frac{3(c-1)}{\alpha}}\cdot\tanh\bigg(\sqrt{\frac{c-1}{2c\beta}}(x-ct)\bigg),\tag{24}$$

$$u_2'=\pm\sqrt{\frac{6(c-1)}{\alpha}}\cdot\mathrm{sech}\bigg(\sqrt{\frac{1-c}{c\beta}}(x-ct)\bigg),\tag{25}$$

$$u_3' = \pm\sqrt{\frac{6(c-1)}{\alpha}}\left[\mathrm{sech}\left(\sqrt{\frac{2(c-1)}{c\beta}}(x-ct)\right) \pm \mathrm{i}\tanh\left(\sqrt{\frac{2(c-1)}{c\beta}}(x-ct)\right)\right]. \tag{26}$$

3.2. Jacobian elliptic functions cn(ξ) *and* dn(ξ) *expansion*

We assume that Eq. (11) has the solution

$$u(\xi) = a_0 + a_1\mathrm{cn}(\xi) + b_1\mathrm{dn}(\xi). \tag{27}$$

Substituting (27) into Eq. (11) yields

$$\begin{aligned}
&\left[(c-1) - \alpha a_0^2 - \alpha\left(1-m^2\right)b_1^2 + k^2c\beta\left(2m^2-1\right)\right]a_1\mathrm{dn}(\xi)\mathrm{sn}(\xi)\\
&+\left[(c-1)m^2 - \alpha a_0^2m^2 - \alpha b_1^2m^2\left(1-m^2\right) - 2\alpha a_1^2\left(1-m^2\right)\right.\\
&\left.-k^2c\beta\left(4m^2-5m^4\right)\right]b_1\mathrm{cn}(\xi)\mathrm{sn}(\xi) - 2\alpha a_0b_1a_1\left(1-m^2\right)\mathrm{sn}(\xi)\\
&-\left(\alpha a_1^2 + 3\alpha b_1^2m^2 + 6k^2c\beta m^2\right)a_1\mathrm{cn}^2(\xi)\mathrm{dn}(\xi)\mathrm{sn}(\xi)\\
&-\left(3\alpha a_1^2 + \alpha b_1^2m^2 + 6k^2c\beta m^2\right)m^2b_1\mathrm{cn}^3(\xi)\mathrm{sn}(\xi)\\
&-2\alpha a_0\left(a_1 + b_1^2m^2\right)\mathrm{cn}(\xi)\mathrm{dn}(\xi)\mathrm{sn}(\xi) - 4\alpha a_0a_1b_1m^2\mathrm{cn}^2(\xi)\mathrm{sn}(\xi) = 0
\end{aligned} \tag{28}$$

from which it is determined that

$$a_0 = 0, \qquad b_1 = 0, \qquad a_1 = \sqrt{\frac{-6k^2c\beta m^2}{\alpha}}, \qquad k^2 = \frac{-(c-1)}{c\beta(2m^2-1)}, \tag{29}$$

$$a_0 = 0, \qquad a_1 = 0, \qquad b_1 = \sqrt{\frac{-6k^2c\beta}{\alpha}}, \qquad k^2 = \frac{1-c}{c\beta(2-m^2)}, \tag{30}$$

$$a_0 = 0, \qquad a_1 = \pm mb_1 = \sqrt{\frac{3(c-1)m^2}{(1+m^2)\alpha}}, \qquad k^2 = \frac{2(1-c)}{(1+m^2)c\beta}. \tag{31}$$

From Eq. (29), we get the same solution as (22), and from (30) and (31), we get

$$u_4 = \pm\sqrt{\frac{6(c-1)}{\alpha(2-m^2)}}\cdot\mathrm{dn}\left(\sqrt{\frac{1-c}{c\beta(2-m^2)}}(x-ct)\right), \tag{32}$$

$$\begin{aligned}
u_5 = &\pm\sqrt{\frac{3(c-1)}{\alpha(1+m^2)}}\cdot\left[m\cdot\mathrm{cn}\left(\sqrt{\frac{2(1-c)}{c\beta(1+m^2)}}(x-ct)\right)\right.\\
&\left.\pm\mathrm{dn}\left(\sqrt{\frac{2(1-c)}{c\beta(1+m^2)}}(x-ct)\right)\right].
\end{aligned} \tag{33}$$

When the modulus $m \to 1$, Eqs. (32) and (33) are reduced to the same solitary solution as (25).

3.3. *Jacobian elliptic functions* sn(ξ) *and* dn(ξ) *expansion*

We assume that the solution of Eq. (11) is

$$u(\xi) \quad = \quad a_0 + a_1 \mathrm{sn}(\xi) + b_1 \mathrm{dn}(\xi). \tag{34}$$

Substituting (34) into Eq. (11) yields

$$\begin{aligned}
&\big[(1-c) + \alpha\big(a_0^2 + b_1^2\big) + k^2 c\beta\big(1+m^2\big)\big] a_1 \mathrm{dn}(\xi)\mathrm{cn}(\xi) \\
&+2\alpha a_0\big(a_1^2 - m^2 b_1^2\big)\mathrm{sn}(\xi)\mathrm{dn}(\xi)\mathrm{cn}(\xi) \\
&+\big[(c-1)m^2 - \alpha\big(a_0^2+b_1^2\big)m^2 + 2\alpha a_1^2 - k^2 c\beta\big(m^4+4m^2\big)\big] b_1 \mathrm{sn}(\xi)\mathrm{cn}(\xi) \\
&-4\alpha a_0 a_1 b_1 m^2 sn^2\xi cn\xi + b_1 m^2\big(\alpha b_1^2 m^2 - 3\alpha a_1^2 + 6k^2 c\beta m^2\big)\mathrm{sn}^3(\xi)\mathrm{cn}(\xi) \\
&-a_1\big(\alpha - 3\alpha b_1^2 m^2 + 6k^2 c\beta m^2\big)\mathrm{sn}^2(\xi)\mathrm{dn}(\xi) cn\xi = 0
\end{aligned} \tag{35}$$

from which it is determined that

$$a_0 = 0, \qquad b_1 = 0, \qquad a_1 = \pm\sqrt{\frac{6k^2 c\beta m^2}{\alpha}}, \qquad k^2 = \frac{c-1}{c\beta(1+m^2)}, \tag{36}$$

$$a_0 = 0, \qquad a_1 = 0, \qquad b_1 = \pm\sqrt{\frac{-6k^2 c\beta}{\alpha}}, \qquad k^2 = \frac{1-c}{c\beta(2-m^2)}, \tag{37}$$

$$a_0 = 0, \qquad a_1 = \pm \mathrm{i} m b_1 = \pm\sqrt{\frac{3m^2k^2c\beta}{2\alpha}}, \qquad k^2 = \frac{2(c-1)}{c\beta(2-m^2)}. \tag{38}$$

From Eqs. (36) and (37), we can get the solutions same as (21) and (32), respectively. From Eqs. (38), we can obtain

$$\begin{aligned}
u_6 \quad = \quad & \pm\sqrt{\frac{3(c-1)}{\alpha(2-m^2)}} \cdot \left[m \cdot \mathrm{sn}\left(\sqrt{\frac{2(c-1)}{c\beta(2-m^2)}}(x-ct)\right)\right. \\
& \left.\pm \mathrm{i} \cdot \mathrm{dn}\left(\sqrt{\frac{2(c-1)}{c\beta(2-m^2)}}(x-ct)\right)\right].
\end{aligned} \tag{39}$$

When the modulus $m \to 1$, Eq. (39) is reduced to the following solitary wave solution

$$u_6' = \pm\sqrt{\frac{3(c-1)}{\alpha}} \cdot \left[\tanh\left(\sqrt{\frac{2(c-1)}{c\beta}}(x-ct)\right) \pm \mathrm{i} \cdot \mathrm{sech}\left(\sqrt{\frac{2(c-1)}{c\beta}}(x-ct)\right)\right]. \tag{40}$$

3.4. *Jacobian elliptic functions* ns(ξ) *and* cs(ξ) *expansion*

We assume that Eq. (11) has the solution

$$u(\xi) = a_0 + a_1 \mathrm{ns}(\xi) + b_1 \mathrm{cs}(\xi). \tag{41}$$

Substituting (41) into Eq. (11) yields

$$\begin{aligned} a_1\big[(c-1) - \alpha\big(a_0^2 - b_1^2\big) - k^2 c\beta\big(1+m^2\big)\big]\mathrm{ns}(\xi)\mathrm{cs}(\xi)\mathrm{dn}(\xi) + 2\alpha a_0 a_1 b_1 \mathrm{ns}(\xi)\mathrm{dn}(\xi) \\ +b_1\big[\big(c-1\big) - \alpha\big(a_0^2 - b_1^2 - 2a_1^2\big) - k^2 c\beta\big(4+m^2\big)\big]\mathrm{ns}^2(\xi)\mathrm{dn}(\xi) \\ -4\alpha a_0 a_1 b_1 \mathrm{ns}^3(\xi)\mathrm{dn}(\xi) - 2\alpha a_0\big(a_1^2 + b_1^2\big)\mathrm{ns}^2(\xi)\mathrm{cs}(\xi)\mathrm{dn}(\xi) \\ -b_1\big(3\alpha a_1^2 + \alpha b_1^2 - 6k^2 c\beta\big)\mathrm{ns}^4(\xi)\mathrm{dn}(\xi) \\ -a_1\big(3\alpha b_1^2 + \alpha a_1^2 - 6k^2 c\beta\big)\mathrm{ns}^3(\xi)\mathrm{cs}(\xi)\mathrm{dn}(\xi) = 0. \end{aligned} \tag{42}$$

From this equation, it is determined that

$$a_0 = 0, \qquad b_1 = 0, \qquad a_1 = \pm\sqrt{\frac{6k^2 c\beta}{\alpha}}, \qquad k^2 = \frac{c-1}{c\beta(1+m^2)}, \tag{43}$$

$$a_0 = 0, \qquad a_1 = 0, \qquad b_1 = \pm\sqrt{\frac{6k^2 c\beta}{\alpha}}, \qquad k^2 = \frac{1-c}{c\beta(2-m^2)}, \tag{44}$$

$$a_0 = 0, \qquad a_1 = \pm b_1 = \sqrt{\frac{3k^2 c\beta}{2\alpha}}, \qquad k^2 = \frac{2(c-1)}{c\beta(2m^2-1)}. \tag{45}$$

Hence, we obtain three new periodic solutions

$$u_7 = \pm\sqrt{\frac{6(c-1)}{\alpha(1+m^2)}} \cdot \mathrm{ns}\left(\sqrt{\frac{c-1}{c\beta(1+m^2)}}(x-ct)\right), \tag{46}$$

$$u_8 = \pm\sqrt{\frac{6(1-c)}{\alpha(2-m^2)}} \cdot \mathrm{cs}\left(\sqrt{\frac{1-c}{c\beta(2-m^2)}}(x-ct)\right), \tag{47}$$

$$\begin{aligned} u_9 = \pm\sqrt{\frac{3(c-1)}{\alpha(2m^2-1)}}\Bigg[\mathrm{ns}\left(\sqrt{\frac{2(c-1)}{c\beta(2m^2-1)}}(x-ct)\right) \\ \pm\mathrm{cs}\left(\sqrt{\frac{2(c-1)}{c\beta(2m^2-1)}}(x-ct)\right)\Bigg]. \end{aligned} \tag{48}$$

When the modulus $m \to 1$, Eqs. (46)–(48) degenerate to the following solitary wave solutions

$$u_7' = \pm\sqrt{\frac{3(c-1)}{\alpha}} \coth\left(\sqrt{\frac{c-1}{2c\beta}}(x-ct)\right), \tag{49}$$

$$u_8' = \pm\sqrt{\frac{6(1-c)}{\alpha}}\mathrm{csch}\left(\sqrt{\frac{1-c}{c\beta}}(x-ct)\right), \tag{50}$$

$$u_9' = \pm\sqrt{\frac{3(c-1)}{\alpha}}\left[\coth\left(\sqrt{\frac{2(c-1)}{c\beta}}(x-ct)\right) \pm\mathrm{csch}\left(\sqrt{\frac{2(c-1)}{c\beta}}(x-ct)\right)\right]. \tag{51}$$

4. *Nonlinear Klein-Gordon equation*

We discuss the following nonlinear Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - c_0^2\frac{\partial^2 u}{\partial x^2} + \alpha u - \beta u^3 = 0. \tag{52}$$

Substituting the traveling solution (2) into Eq. (52) yields

$$k^2(c^2 - c_0^2)\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + \alpha u - \beta u^3 = 0. \tag{53}$$

It is easily seen that

$$n = 1. \tag{54}$$

By using different double Jacobian elliptic functions expansion, we can obtain the following periodic solutions

$$u_1 = \pm\sqrt{\frac{2m^2\alpha}{\beta(1+m^2)}}\cdot\mathrm{sn}\left(\sqrt{\frac{\alpha}{(1+m^2)(c^2-c_0^2)}}(x-ct)\right), \tag{55}$$

$$u_2 = \pm\sqrt{\frac{2m^2\alpha}{\beta(2m^2-1)}}\cdot\mathrm{cn}\left(\sqrt{\frac{\alpha}{(2m^2-1)(c_0^2-c^2)}}(x-ct)\right), \tag{56}$$

$$u_3 = \pm\sqrt{\frac{-m^2\alpha}{\beta(2-m^2)}}\cdot\left[\mathrm{sn}\left(\sqrt{\frac{2\alpha}{(2-m^2)(c^2-c_0^2)}}(x-ct)\right) \pm\mathrm{i}\cdot\mathrm{cn}\left(\sqrt{\frac{2\alpha}{(2-m^2)(c^2-c_0^2)}}(x-ct)\right)\right], \tag{57}$$

$$u_4 = \pm\sqrt{\frac{2\alpha}{\beta(2-m^2)}}\cdot\mathrm{dn}\left(\sqrt{\frac{\alpha}{(c_0^2-c^2)(2-m^2)}}(x-ct)\right), \tag{58}$$

$$u_5 = \pm\sqrt{\frac{\alpha}{\beta(2m^2-1)}}\Bigg[m\cdot\mathrm{sn}\left(\sqrt{\frac{2\alpha}{(c^2-c_0^2)(2m^2-1)}}(x-ct)\right) \pm\mathrm{i}\cdot\mathrm{dn}\left(\sqrt{\frac{2\alpha}{(c^2-c_0^2)(2m^2-1)}}(x-ct)\right)\Bigg], \tag{59}$$

$$u_6 = \pm\sqrt{\frac{\alpha}{\beta(1+m^2)}}\Bigg[m\cdot\mathrm{cn}\left(\sqrt{\frac{2\alpha}{(1+m^2)(c_0^2-c^2)}}(x-ct)\right) \pm\mathrm{dn}\left(\sqrt{\frac{2\alpha}{(1+m^2)(c_0^2-c^2)}}(x-ct)\right)\Bigg], \tag{60}$$

$$u_7 = \pm\sqrt{\frac{2\alpha}{(1+m^2)\beta}}\cdot\mathrm{ns}\left(\sqrt{\frac{\alpha}{(1+m^2)(c^2-c_0^2)}}(x-ct)\right), \tag{61}$$

$$u_8 = \pm\sqrt{\frac{-2\alpha}{(2-m^2)\beta}}\cdot\mathrm{cs}\left(\sqrt{\frac{\alpha}{(2-m^2)(c_0^2-c^2)}}(x-ct)\right), \tag{62}$$

$$u_9 = \pm\sqrt{\frac{-\alpha}{(2m^2-1)\beta}}\Bigg[\mathrm{ns}\left(\sqrt{\frac{2\alpha}{(c^2-c_0^2)(2m^2-1)}}(x-ct)\right) \pm\mathrm{cs}\left(\sqrt{\frac{2\alpha}{(c^2-c_0^2)(2m^2-1)}}(x-ct)\right)\Bigg]. \tag{63}$$

When the modulus $m \to 1$, we can obtain the following solitary solutions

$$u_1' = \pm\sqrt{\frac{\alpha}{\beta}}\tanh\left(\sqrt{\frac{\alpha}{2(c^2-c_0^2)}}(x-ct)\right), \tag{64}$$

$$u_2' = \pm\sqrt{\frac{2\alpha}{\beta}}\mathrm{sech}\left(\sqrt{\frac{\alpha}{c_0^2-c^2}}(x-ct)\right), \tag{65}$$

$$u_3' = \pm\sqrt{\frac{-\alpha}{\beta}}\left[\tanh\left(\sqrt{\frac{\alpha}{c^2-c_0^2}}(x-ct)\right) \pm \mathrm{i}\ \mathrm{sech}\left(\sqrt{\frac{\alpha}{c^2-c_0^2}}(x-ct)\right)\right], \tag{66}$$

$$u_4' = \pm\sqrt{\frac{\alpha}{\beta}}\left[\tanh\left(\sqrt{\frac{\alpha}{c^2-c_0^2}}(x-ct) \pm \mathrm{i}\ \mathrm{sech}\left(\sqrt{\frac{\alpha}{c^2-c_0^2}}(x-ct)\right)\right], \tag{67}$$

$$u_5' = \pm\sqrt{\frac{\alpha}{\beta}}\cdot\coth\left(\sqrt{\frac{\alpha}{2(c^2-c_0^2)}}(x-ct)\right), \tag{68}$$

$$u_6' = \pm\sqrt{\frac{-2\alpha}{\beta}}\cdot\mathrm{csch}\left(\sqrt{\frac{\alpha}{c_0^2-c^2}}(x-ct)\right), \tag{69}$$

$$u'_7 = \pm\sqrt{\frac{-\alpha}{\beta}} \cdot \left[\coth\left(\sqrt{\frac{2\alpha}{c^2-c_0^2}}(x-ct)\right) \pm \mathrm{csch}\left(\sqrt{\frac{2\alpha}{c^2-c_0^2}}(x-ct)\right)\right]. \quad (70)$$

5. *Conclusion*

In this paper, the new extended Jacobian elliptic function expansion method is applied to BBM equation and nonlinear Klein-Gordon equation. Many different periodic solutions are obtained by this method based on different double Jacobian elliptic function expansion. Also many solitary solutions are obtained in the limiting conditions, which shows that this method is powerful. We believe that if the rank of every term in Eq. (3) is even or odd, this method can be used to construct explicit solutions.

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POOPĆENA METODA RAZVOJA PO JACOBIJEVIM ELIPTIČKIM FUNKCIJAMA I PRIMJENE NA NELINEARNE VALNE JEDNADŽBE

Primijenili smo proširenu metodu razvoja po Jacobijevim eliptičkim funkcijama za izvod točnih periodičnih rješenja dviju nelinearnih valnih jednadžbi. Periodična rješenja koja smo izveli tom metodom svode se na solitonska rješenja u određenim graničnim uvjetima.